Part III — Positivity in Algebraic Geometry
Theorems with proof

Based on lectures by S. Svaldi
Notes taken by Dexter Chua

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This class aims at giving an introduction to the theory of divisors, linear systems and their positivity properties on projective algebraic varieties.

The first part of the class will be dedicated to introducing the basic notions and results regarding these objects and special attention will be devoted to discussing examples in the case of curves and surfaces.

In the second part, the course will cover classical results from the theory of divisors and linear systems and their applications to the study of the geometry of algebraic varieties.

If time allows and based on the interests of the participants, there are a number of more advanced topics that could possibly be covered: Reider's Theorem for surfaces, geometry of linear systems on higher dimensional varieties, multiplier ideal sheaves and invariance of plurigenera, higher dimensional birational geometry.

Pre-requisites

The minimum requirement for those students wishing to enroll in this class is their knowledge of basic concepts from the Algebraic Geometry Part 3 course, i.e. roughly Chapters 2 and 3 of Hartshorne's Algebraic Geometry.

Familiarity with the basic concepts of the geometry of algebraic varieties of dimension 1 and 2 — e.g. as covered in the preliminary sections of Chapters 4 and 5 of Hartshorne's Algebraic Geometry — would be useful but will not be assumed — besides what was already covered in the Michaelmas lectures.

Students should have also some familiarity with concepts covered in the Algebraic Topology Part 3 course such as cohomology, duality and characteristic classes.

Contents

1 Divisors

1.1 Projective embeddings

Theorem. Let A be any ring, and X a scheme over A.

(i) If $\varphi : X \to \mathbb{P}^n$ is a morphism over A, then $\varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$ is an invertible sheaf on X, generated by the sections $\varphi^*x_0, \ldots, \varphi^*x_n \in H^0(X, \varphi^*\mathcal{O}_{\mathbb{P}^n}(1))$.

(ii) If $\mathcal{L}$ is an invertible sheaf on X, and if $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ which generate $\mathcal{L}$, then there exists a unique morphism $\varphi : X \to \mathbb{P}^n$ such that $\varphi^*\mathcal{O}(1) \cong \mathcal{L}$ and $\varphi^*x_i = s_i$.

Proof.

(i) The pullback of an invertible sheaf is an invertible, and the pullbacks of $x_0, \ldots, x_n$ generate $\varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$.

(ii) In short, we map $x \in X$ to $[s_0(x) : \cdots : s_n(x)] \in \mathbb{P}^n$.

In more detail, define

$$X_{s_i} = \{p \in X : s_i \not\in \mathfrak{m}_p\mathcal{L}_p\}.$$

This is a Zariski open set, and s_i is invertible on X_{s_i}. Thus there is a dual section $s_i^\vee \in \mathcal{L}^\vee$ such that $s_i \otimes s_i^\vee \in \mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$ is equal to 1. Define the map $X_{s_i} \to \mathbb{A}^n$ by the map

$$\begin{aligned} K[\mathbb{A}^n] &\to H^0(X_{s_i}, \mathcal{O}_{s_i}) \\ y_i &\mapsto s_j \otimes s_i^\vee. \end{aligned}$$

Since the s_i generate, they cannot simultaneously vanish on a point. So $X = \bigcup X_{s_i}$. Identifying $\mathbb{A}^n$ as the chart of $\mathbb{P}^n$ where $x_i \neq 0$, this defines the desired map $X \to \mathbb{P}^n$. □

Proposition. Let $K = \bar{K}$, and X a projective variety over K. Let $\mathcal{L}$ be an invertible sheaf on X, and $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ generating sections. Write $V = \langle s_0, \ldots, s_n \rangle$ for the linear span. Then the associated map $\varphi : X \to \mathbb{P}^n$ is a closed embedding iff

(i) For every distinct closed points $p \neq q \in X$, there exists $s_{p,q} \in V$ such that $s_{p,q} \in \mathfrak{m}_p\mathcal{L}_p$ but $s_{p,q} \not\in \mathfrak{m}_q\mathcal{L}_q$.

(ii) For every closed point $p \in X$, the set $\{s \in V \mid s \in \mathfrak{m}_p\mathcal{L}_p\}$ spans the vector space $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$.

Proof.

($\Rightarrow$) Suppose ϕ is a closed immersion. Then it is injective on points. So suppose $p \neq q$ are (closed) points. Then there is some hyperplane $H_{p,q}$ in $\mathbb{P}^n$ passing through p but not q. The hyperplane $H_{p,q}$ is the vanishing locus of a global section of $\mathcal{O}(1)$. Let $s_{p,q} \in V \subseteq H^0(X, \mathcal{L})$ be the pullback of this section. Then $s_{p,q} \in \mathfrak{m}_p\mathcal{L}_p$ and $s_{p,q} \not\in \mathfrak{m}_q\mathcal{L}_q$. So (i) is satisfied.

To see (ii), we restrict to the affine patch containing p, and X is a closed subvariety of $\mathbb{A}^n$. The result is then clear since $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$ is exactly the span of $s_0, \ldots, s_n$. We used $K = \bar{K}$ to know what the closed points of $\mathbb{P}^n$ look like.

($\Leftarrow$) We first show that φ is injective on closed points. For any $p \neq q \in X$, write the given $s_{p,q}$ as

$$s_{p,q} = \sum \lambda_i s_i = \sum \lambda_i \varphi^* x_i = \varphi^* \sum \lambda_i x_i$$

for some $\lambda_i \in K$. So we can take $H_{p,q}$ to be given by the vanishing set of $\sum \lambda_i x_i$, and so it is injective on closed point. It follows that it is also injective on schematic points. Since X is proper, so is φ, and in particular φ is a homeomorphism onto the image.

To show that φ is in fact a closed immersion, we need to show that $\mathcal{O}_{\mathbb{P}^n} \to \varphi_* \mathcal{O}_X$ is surjective. As before, it is enough to prove that it holds at the level of stalks over closed points. To show this, we observe that $\mathcal{L}_p$ is trivial, so $\mathfrak{m}_p \mathcal{L}_p / \mathfrak{m}_p^2 \mathcal{L}_p \cong \mathfrak{m}_p / \mathfrak{m}_p^2$ (unnaturally). We then apply the following lemma:

Lemma. Let $f : A \to B$ be a local morphism of local rings such that

- $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism;
- $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective; and
- B is a finitely-generated A-module.

Then f is surjective. □

To check the first condition, note that we have

$$\frac{\mathcal{O}_{p,\mathbb{P}^n}}{\mathfrak{m}_{p,\mathbb{P}^n}} \cong \frac{\mathcal{O}_{p,X}}{\mathfrak{m}_{p,X}} \cong K.$$

Now since $\mathfrak{m}_{p,\mathbb{P}^n}$ is generated by $x_0, \ldots, x_n$, the second condition is the same as saying

$$\mathfrak{m}_{p,\mathbb{P}^n} \to \frac{\mathfrak{m}_{p,X}}{\mathfrak{m}_{p,X}^2}$$

is surjective. The last part is immediate.

Theorem (Serre)**.** Let X be a projective scheme over a Noetherian ring A, $\mathcal{L}$ be a very ample invertible sheaf, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then there exists a positive integer $n_0 = n_0(\mathcal{F}, \mathcal{L})$ such that for all $n \geq n_0$, the twist $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. □

Theorem (Serre)**.** Let X be a scheme of finite type over a Noetherian ring A, and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample iff there exists $m > 0$ such that $\mathcal{L}^m$ is very ample.

Proof.

($\Leftarrow$) Let $\mathcal{L}^m$ be very ample, and $\mathcal{F}$ a coherent sheaf. By Serre's theorem, there exists n_0 such that for all $j \geq j_0$, the sheafs

$$\mathcal{F} \otimes \mathcal{L}^{mj}, (\mathcal{F} \otimes \mathcal{L}) \otimes \mathcal{L}^{mj}, \ldots, (\mathcal{F} \otimes \mathcal{L}^{m-1}) \otimes \mathcal{L}^{mj}$$

are all globally generated. So $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for $n \geq mj_0$.

(⇒) Suppose $\mathcal{L}$ is ample. Then $\mathcal{L}^m$ is globally generated for m sufficiently large. We claim that there exists $t_1, \ldots, t_n \in H^0(X, \mathcal{L}^N)$ such that $\mathcal{L}|_{X_{t_i}}$ are all trivial (i.e. isomorphic to $\mathcal{O}_{X_{t_i}}$), and $X = \bigcup X_{t_i}$.

By compactness, it suffices to show that for each $p \in X$, there is some $t \in H^0(X, \mathcal{L}^n)$ (for some n) such that $p \in X_{t_i}$ and $\mathcal{L}$ is trivial on X_{t_i}. First of all, since $\mathcal{L}$ is locally free by definition, we can find an open affine U containing p such that $\mathcal{L}|_U$ is trivial.

Thus, it suffices to produce a section t that vanishes on $Y = X - U$ but not at p. Then $p \in X_t \subseteq U$ and hence $\mathcal{L}$ is trivial on X_t. Vanishing on Y is the same as belonging to the ideal sheaf $\mathcal{I}_Y$. Since $\mathcal{I}_Y$ is coherent, ampleness implies there is some large n such that $\mathcal{I}_Y \otimes \mathcal{L}^n$ is generated by global sections. In particular, since $\mathcal{I}_Y \otimes \mathcal{L}^n$ doesn't vanish at p, we can find some $t \in \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^N)$ such that $t \not\in \mathfrak{m}_p(\mathcal{I}_Y \otimes \mathcal{L}^n)_p$. Since $\mathcal{I}_Y$ is a subsheaf of $\mathcal{O}_X$, we can view t as a section of $\mathcal{L}^n$, and this t works.

Now given the X_{t_i}, for each fixed i, we let $\{b_{ij}\}$ generate $\mathcal{O}_{X_{t_i}}$ as an A-algebra. Then for large n, $c_{ij} = t_i^n b_{ij}$ extends to a global section $c_{ij} \in \Gamma(X, \mathcal{L}^n)$ (by clearing denominators). We can pick an n large enough to work for all b_{ij}. Then we use $\{t_i^n, c_{ij}\}$ as our generating sections to construct a morphism to $\mathbb{P}^N$, and let $\{x_i, x_{ij}\}$ be the corresponding coordinates. Observe that $\bigcup X_{t_i} = X$ implies the t_i^n already generate $\mathcal{L}^n$. Now each x_{t_i} gets mapped to $U_i \subseteq \mathbb{P}^N$, the vanishing set of x_i. The map $\mathcal{O}_{U_i} \to \varphi_* \mathcal{O}_{X_{t_i}}$ corresponds to the map

$$A[y_i, y_{ij}] \to \mathcal{O}_{X_{t_i}},$$

where y_{ij} is mapped to $c_{ij}/t_i^n = b_{ij}$. So by assumption, this is surjective, and so we have a closed embedding. □

Proposition. Let $\mathcal{L}$ be a sheaf over X (which is itself a projective variety over K). Then the following are equivalent:

(i) $\mathcal{L}$ is ample.

(ii) $\mathcal{L}^m$ is ample for all $m > 0$.

(iii) $\mathcal{L}^m$ is ample for some $m > 0$. □

Theorem (Serre)**.** Let X be a projective scheme over a Noetherian ring A, and $\mathcal{L}$ is very ample on X. Let $\mathcal{F}$ be a coherent sheaf. Then

(i) For all $i \geq 0$ and $n \in \mathbb{N}$, $H^i(\mathcal{F} \otimes \mathcal{L}^n)$ is a finitely-generated A-module.

(ii) There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $H^i(\mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $i > 0$. □

Theorem. Let X be a proper scheme over a Noetherian ring A, and $\mathcal{L}$ an invertible sheaf. Then the following are equivalent:

(i) $\mathcal{L}$ is ample.

(ii) For all coherent $\mathcal{F}$ on X, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $H^i(\mathcal{F} \otimes \mathcal{L}^n) = 0$.

Proof. Proving (i) $\Rightarrow$ (ii) is the same as the first part of the theorem last time.

To prove (ii) $\Rightarrow$ (i), fix a point $x \in X$, and consider the sequence

$$0 \to \mathfrak{m}_x \mathcal{F} \to \mathcal{F} \to \mathcal{F}_x \to 0.$$

We twist by $\mathcal{L}^n$, where n is sufficiently big, and take cohomology. Then we have a long exact sequence

$$0 \to H^0(\mathfrak{m}_x \mathcal{F}(n)) \to H^0(\mathcal{F}(n)) \to H^0(\mathcal{F}_x(n)) \to H^1(\mathfrak{m}_x \mathcal{F}(n)) = 0.$$

In particular, the map $H^0(\mathcal{F}(n)) \to H^0(\mathcal{F}_x(n))$ is surjective. This mean at x, $\mathcal{F}(n)$ is globally generated. Then by compactness, there is a single n large enough such that $\mathcal{F}(n)$ is globally generated everywhere. □

1.2 Weil divisors

Theorem (Hartog's lemma)**.** Let X be normal, and $f \in \mathcal{O}(X \setminus V)$ for some $V \geq 2$. Then $f \in \mathcal{O}_X$. Thus, $\operatorname{div}(f) = 0$ implies $f \in \mathcal{O}_X^\times$.

1.3 Cartier divisors

Proposition. If X is normal, then

$$\operatorname{div} : \{\text{rational sections of } \mathcal{L}\} \to \operatorname{WDiv}(X).$$

is well-defined, and two sections have the same image iff they differ by an element of $\mathcal{O}_X^*$.

Corollary. If X is normal and proper, then there is a map

$$\operatorname{div}\{\text{rational sections of } \mathcal{L}\}/K^* \to \operatorname{WDiv}(X).$$

Proof. Properness implies $\mathcal{O}_X^* = K^*$. □

Proposition. $\mathcal{O}_X(D)$ is a rank 1 quasicoherent $\mathcal{O}_X$-module. □.

Proposition. If D is locally principal at every point x, then $\mathcal{O}_X(D)$ is an invertible sheaf.

Proof. If $U \subseteq X$ is such that $D|_U = \operatorname{div}(f)|_U$, then there is an isomorphism

$$\begin{aligned} \mathcal{O}_X|_U &\to \mathcal{O}_X(D)|_U \\ g &\mapsto g/f. \end{aligned}$$

□

Proposition. If D_1, D_2 are Cartier divisors, then

(i) $\mathcal{O}_X(D_1 + D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$.

(ii) $\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^\vee$.

(iii) If $f \in K(X)$, then $\mathcal{O}_X(\operatorname{div}(f)) \cong \mathcal{O}_X$. □

Proposition. Let X be a Noetherian, normal, integral scheme. Assume that X is *factorial*, i.e. every local ring $\mathcal{O}_{X,x}$ is a UFD. Then any Weil divisor is Cartier.

Proof. It is enough to prove the proposition when D is prime and effective. So $D \subseteq X$ is a codimension 1 irreducible subvariety. For $x \in D$

- If $x \notin D$, then 1 is a divisor equivalent to D near x.
- If $x \in D$, then $I_{D,x} \subseteq \mathcal{O}_{X,x}$ is a height 1 prime ideal. So $I_{D,x} = (f)$ for $f \in \mathfrak{m}_{X,x}$. Then f is the local equation for D. □

Theorem. Let X be normal and $\mathcal{L}$ an invertible sheaf, s a rational section of $\mathcal{L}$. Then $\mathcal{O}_X(\mathrm{div}(s))$ is invertible, and there is an isomorphism

$$\mathcal{O}_X(\mathrm{div}(s)) \to \mathcal{L}.$$

Moreover, sending $\mathcal{L}$ to $\mathrm{div} s$ gives a map

$$\mathrm{Pic}(X) \to \mathrm{Cl}(X),$$

which is an isomorphism if X is factorial (and Noetherian and integral).

Proof. Given $f \in H^0(U, \mathcal{O}_X(\mathrm{div}(s)))$, map it to $f \cdot s \in H^0(U, \mathcal{L})$. This gives the desired isomorphism.

If we have to sections $s' \neq s$, then $f = s'/s \in K(X)$. So $\mathrm{div}(s) = \mathrm{div}(s') + \mathrm{div}(f)$, and $\mathrm{div}(f)$ is principal. So this gives a well-defined map $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$. □

1.4 Computations of class groups

Proposition. Let X be an integral scheme, regular in codimension 1. If $Z \subseteq X$ is an integral closed subscheme of codimension 1, then we have an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus Z) \to 0,$$

where $n \in \mathbb{Z}$ is mapped to $[nZ]$.

Proof. The map $\mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus Z)$ is given by restriction. If S is a Weil divisor on $X \setminus Z$, then $\bar{S} \subseteq X$ maps to S under the restriction map. So this map is surjective.

Also, that $[nZ]|_{X \setminus Z}$ is trivial. So the composition of the first two maps vanishes. To check exactness, suppose D is a Weil divisor on X, principal on $X \setminus Z$. Then $D|_{X \setminus Z} = \dim(f)|_{X \setminus Z}$ for some $f \in K(X)$. Then $D - \mathrm{div}(f)$ is just supported along Z. So it must be of the form nZ. □

Proposition. If $Z \subseteq X$ has codimension ≥ 2, then $\mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus Z)$ is an isomorphism.

Proposition. If A is a Noetherian ring, regular in codimension 1, then A is a UFD iff A is normal and $\mathrm{Cl}(\mathrm{Spec}\, A) = 0$

Proof. If A is a UFD, then it is normal, and every prime ideal of height 1 is principally generated. So if $D \in \mathrm{Spec}\, A$ is Weil and prime, then $D = V(f)$ for some f, and hence $(f) = I_D$.

Conversely, if A is normal and $\mathrm{Cl}(\mathrm{Spec}\, A) = 0$, then every Weil divisor is principal. So if I is a height 1 prime ideal, then $V(I) = D$ for some Weil divisor D. Then D is principal. So $I = (f)$ for some f. So A is a Krull Noetherian integral domain with principally generated height 1 prime ideals. So it is a UFD. □

Proposition. Let X be Noetherian and regular in codimension one. Then

$$\mathrm{Cl}(X) = \mathrm{Cl}(X \times \mathbb{A}^1).$$

Proof. We have a projection map

$$\begin{aligned} \mathrm{pr}_1^* : \mathrm{Cl}(X) &\to \mathrm{Cl}(X \times \mathbb{A}^1) \\ [D_i] &\mapsto [D_i \times \mathbb{A}^1] \end{aligned}$$

It is an exercise to show that is injective. ‘

To show surjectivity, first note that we can the previous exact sequence and the 4-lemma to assume X is affine.

We consider what happens when we localize a prime divisor D at the generic point of X. Explicitly, suppose $\mathcal{I}_D$ is the ideal of D in $K[X \times \mathbb{A}^1]$, and let $\mathcal{I}_D^0$ be the ideal of $K(X)[t]$ generated b $\mathcal{I}_D$ under the inclusion

$$K[X \times \mathbb{A}^1] = K[X][t] \subseteq K(X)[t].$$

If $\mathcal{I}_D^0 = 1$, then $\mathcal{I}_D$ contains some function $f \in K(X)$. Then $D \subseteq V(f)$ as a subvariety of $X \times \mathbb{A}^1$. So D is an irreducible component of $V(f)$, and in particular is of the form $D' \times \mathbb{A}^1$.

If not, then $\mathcal{I}_D^0 = (f)$ for some $f \in K(X)[t]$, since $K(X)[t]$ is a PID. Then $\mathrm{div} f$ is a principal divisor of $X \times \mathbb{A}^1$ whose localization at the generic point is D. Thus, $\mathrm{div} f$ is D plus some other divisors of the form $D' \times \mathbb{A}^1$. So D is linearly equivalent to a sum of divisors of the form $D' \times \mathbb{A}^1$. □

1.5 Linear systems

Proposition. Let X be a smooth projective variety over an algebraically closed field. Let D_0 be a divisor on X.

(i) For all $s \in H^0(X, \mathcal{O}_X(D))$, $\mathrm{div}(s)$ is an effective divisor linearly equivalent to D.

(ii) If $D \sim D_0$ and $D \geq 0$, then there is $s \in H^0(\mathcal{O}_X(D_0))$ such that $\mathrm{div}(s) = D$

(iii) If $s, s' \in H^0(\mathcal{O}_X(D_0))$ and $\mathrm{div}(s) = \mathrm{div}(s')$, then $s' = \lambda s$ for some $\lambda \in K^*$.

Proof.

(i) Done last time.

(ii) If $D \sim D_0$, then $D - D_0 = \mathrm{div}(f)$ for some $f \in K(X)$. Then $(f) + D_0 \geq 0$. So f induces a section $s \in H^0(\mathcal{O}_X(D_0))$. Then $\mathrm{div}(s) = D$.

(iii) We have $\frac{s'}{s} \in K(X)^*$. So $\mathrm{div}\left(\frac{s'}{s}\right)$. So $\frac{s'}{s} \in H^0(\mathcal{O}^*) = K^*$. □

Theorem (Riemann–Roch theorem)**.** If C is a smooth projective curve, then

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g(C).$$ □

Proposition. Let D be a Cartier divisor on a projective normal scheme .Then $D \sim H_1 - H_2$ for some very ample divisors H_i. We can in fact take H_i to be effective, and if X is smooth, then we can take H_i to be smooth and intersecting transversely. □

Theorem (Bertini)**.** Let X be a smooth projective variety over an algebraically closed field K, and D a very ample divisor. Then there exists a Zariski open set $U \subseteq |D|$ such that for all $H \in U$, H is smooth on X and if $H_1 \neq H_2$, then H_1 and H_2 intersect transversely. □

2 Surfaces

2.1 The intersection product

Proposition.

(i) The product $D_1 \cdot D_2$ depends only on the classes of D_1, D_2 in $\operatorname{Pic}(X)$.

(ii) $D_1 \cdot D_2 = D_2 \cdot D_1$.

(iii) $D_1 \cdot D_2 = |D_1 \cap D_2|$ if D_1 and D_2 are curves intersecting transversely.

(iv) The intersection product is bilinear.

Proof. Only (iv) requires proof. First observe that if H is a very ample divisor represented by a smooth curve, then we have

$$H \cdot D = \deg_H(\mathcal{O}_H(D)),$$

and this is linear in D.

Next, check that $D_1 \cdot (D_2 + D_3) - D_1 \cdot D_2 - D_1 \cdot D_3$ is symmetric in D_1, D_2, D_3. So

(a) Since this vanishes when D_1 is very ample, it also vanishes if D_2 or D_3 is very ample.

(b) Thus, if H is very ample, then $D \cdot (-H) = -(D \cdot H)$.

(c) Thus, if H is very ample, then $(-H) \cdot D$ is linear in D.

(d) If D is any divisor, write $D = H_1 - H_2$ for H_1, H_2 very ample and smooth. Then $D \cdot D' = H_1 \cdot D' - H_2 \cdot D'$ by (a), and thus is linear in D'. □

Theorem (Riemann–Roch for surfaces)**.** Let $D \in \operatorname{Div}(X)$. Then

$$\chi(X, \mathcal{O}_X(D)) = \frac{D \cdot (D - K_X)}{2} + \chi(\mathcal{O}_X),$$

where K_X is the *canonical divisor*.

Theorem (Adjunction formula)**.** Let X be a smooth surface, and $C \subseteq X$ a smooth curve. Then

$$(\mathcal{O}_X(K_X) \otimes \mathcal{O}_X(C))|_C \cong \mathcal{O}_C(K_C).$$

Proof. Let $\mathcal{I}_C = \mathcal{O}_X(-C)$ be the ideal sheaf of C. We then have a short exact sequence on C:

$$0 \to \mathcal{O}_X(-C)|_C \cong \mathcal{I}_C/\mathcal{I}_C^2 \to \Omega^1_X|_C \to \Omega^1_C \to 0,$$

where the left-hand map is given by d. To check this, note that locally on affine charts, if C is cut out by the function f, then smoothness of C implies the kernel of the second map is the span of $\mathrm{d}f$.

By definition of the canonical divisor, we have

$$\mathcal{O}_X(K_X) = \det(\Omega^1_X).$$

Restricting to C, we have

$$\mathcal{O}_X(K_X)|_C = \det(\Omega^1_X|_C) = \det(\mathcal{O}_X(-C)|_C) \otimes \det(\Omega^1_C) = \mathcal{O}_X(C)|_C^\vee \otimes \mathcal{O}_C(K_C).$$

□

Proof of Riemann–Roch. We can assume $D = H_1 - H_2$ for very ample line bundles H_1, H_2, which are smoothly irreducible curves that intersect transversely. We have short exact sequences

$$0 \longrightarrow \mathcal{O}_X(H_1 - H_2) \longrightarrow \mathcal{O}_X(H_1) \longrightarrow \mathcal{O}_{H_2}(H_1|_{H_2}) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(H_1) \longrightarrow \mathcal{O}_{H_1}(H_1) \longrightarrow 0$$

where $\mathcal{O}_{H_1}(H_1)$ means the restriction of the *line bundle* $\mathcal{O}_X(H_1)$ to H_1. We can then compute

$$\begin{aligned}\chi(H_1 - H_2) &= \chi(\mathcal{O}_X(H_1)) - \chi(H_2, \mathcal{O}_{H_2}(H_1|_{H_2})) \\ &= \chi(\mathcal{O}_X) + \chi(H_1, \mathcal{O}_{H_1}(H_1)) - \chi(H_2, \mathcal{O}_{H_2}(H_1|_{H_2})).\end{aligned}$$

The first term appears in our Riemann–Roch theorem, so we leave it alone. We then use Riemann–Roch for curves to understand the remaining. It tells us

$$\chi(H_i, \mathcal{O}_{H_i}(H_1)) = \deg(\mathcal{O}_{H_i}(H_1)) + 1 - g(H_i) = (H_i \cdot H_1) + 1 - g(H_i).$$

By definition of genus, we have

$$2g(H_i) - 2 = \deg(K_{H_i}).$$

and the adjunction formula lets us compute $\deg(K_{H_i})$ by

$$\deg(K_{H_i}) = H_i \cdot (K_X + H_i).$$

Plugging these into the formula and rearranging gives the desired result. □

Theorem (Hodge index theorem)**.** Let X be a projective surface, and H be a (very) ample divisor on X. Let D be a divisor on X such that $D \cdot H = 0$ but $D \not\equiv 0$. Then $D^2 < 0$.

Proof. Before we begin the proof, observe that if H' is very ample and D' is (strictly) effective, then $H' \cdot D' > 0$, since this is given by the number of intersections between D' and any hyperplane in the projective embedding given by H'.

Now assume for contradiction that $D^2 \geq 0$.

- If $D^2 > 0$, fix an n such that $H_n = D + nH$ is very ample. Then $H_n \cdot D > 0$ by assumption.

  We use Riemann–Roch to learn that

  $$\chi(X, \mathcal{O}_X(mD)) = \frac{m^2 D^2 - m K_X \cdot D}{2} + \chi(\mathcal{O}_X).$$

  We first consider the $H^2(\mathcal{O}_X(mD))$ term in the left-hand side. By Serre duality, we have

  $$H^2(\mathcal{O}_X(mD)) = H^0(K_X - mD).$$

  Now observe that for large m, we have

  $$H_n \cdot (K_X - mD) < 0.$$

Since H_n is very ample, it cannot be the case that $K_X - mD$ is effective. So $H^0(K_X - D) = 0$.

Thus, for m sufficiently large, we have

$$h^0(mD) - h^1(mD) > 0.$$

In particular, mD is (strictly) effective for $m \gg 0$. But then $H \cdot mD > 0$ since H is very ample and mD is effective. This is a contradiction.

– If $D^2 = 0$. Since D is not numerically trivial, there exists a divisor E on X such that $D \cdot E \neq 0$. We define

$$E' = (H^2)E - (E \cdot H)H.$$

It is then immediate that $E' \cdot H = 0$. So $D'_n = nD + E'$ satisfies $D'_n \cdot H = 0$. On the other hand,

$$(D'_n)^2 = (E')^2 + 2nD \cdot E' > 0$$

for n large enough. This contradicts the previous part. □

2.2 Blow ups

Lemma.

$$\pi^* C = \tilde{C} + mE,$$

where m is the multiplicity of C at p.

Proof. Choose local coordinates x, y at p and suppose the curve $y = 0$ is not tangent to any branch of C at p. Then in the local ring $\hat{\mathcal{O}}_{X,p}$, the equation of C is given as

$$f = f_m(x, y) + \text{higher order terms},$$

where f_m is a non-zero degree m polynomial. Then by definition, the multiplicity is m. Then on $\bar{U} \subseteq (U \times \mathbb{P}^1)$, we have the chart $U \times \mathbb{A}^1$ where $X \neq 0$, with coordinates

$$(x, y, Y/X = t).$$

Taking (x, t) as local coordinates, the map to U is given by $(x, t) \mapsto (x, xt)$. Then

$$\varepsilon^* f = f(x, tx) = x^m f_m(1, t) + \text{higher order terms} = x^m \cdot h(x, t),$$

with $h(0, 0) \neq 0$. But then on $\bar{U}_{x \neq 0}$, the curve $x = 0$ is just E. So this equation has multiplicity m along E. □

Proposition. Let X be a smooth projective surface, and $x \in X$. Let $\bar{X} = \mathrm{Bl}_x X \overset{\pi}{\to} X$. Then

(i) $\pi^* \mathrm{Pic}(X) \oplus \mathbb{Z}[E] = \mathrm{Pic}(\bar{X})$

(ii) $\pi^* D \cdot \pi^* F = D \cdot F, \quad \pi^* D \cdot E = 0, \quad E^2 = -1.$

(iii) $K_{\bar{X}} = \pi^*(K_X) + E$.

(iv) π^* is defined on $\mathrm{NS}(X)$. Thus,

$$\mathrm{NS}(\bar{X}) = \mathrm{NS}(X) \oplus \mathbb{Z}[E].$$

Proof.

(i) Recall we had the localization sequence

$$\mathbb{Z} \to \mathrm{Pic}(\bar{X}) \to \mathrm{Pic}(X) \to 1$$

and there is a right splitting. To show the result, we need to show the left-hand map is injective, or equivalently, $mE \not\sim 0$. This will follow from the fact that $E^2 = -1$.

(ii) For the first part, it suffices to show that $\pi^* D \cdot \pi^* F = D \cdot F$ for D, F very ample, and so we may assume D, F are curves not passing through x. Then their pullbacks is just the pullback of the curve, and so the result is clear. The second part is also clear.

Finally, pick a smooth curve C passing through E with multiplicity 1. Then

$$\pi^* C = \tilde{C} + E.$$

Then we have

$$0 = \pi^* C \cdot E = \tilde{C} \cdot E + E^2,$$

But $\tilde{C}$ and E intersect at exactly one point. So $\tilde{C} \cdot E = 1$.

(iii) By the adjunction formula, we have

$$(K_{\bar{X}} + E) \cdot E = \deg(K_{\mathbb{P}^1}) = -2.$$

So we know that $K_{\bar{X}} \cdot E = -1$. Also, outside of E, $K_{\bar{X}}$ and K_X agree. So we have

$$K_{\bar{X}} = \pi^*(K_X) + mE$$

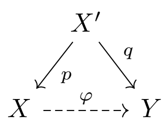
for some $m \in \mathbb{Z}$. Then computing

$$-1 = K_{\bar{X}} \cdot E = \pi^*(K_X) \cdot E + mE^2$$

gives $m = 1$.

(iv) We need to show that if $D \in \mathrm{Num}_0(X)$, then $\pi^*(D) \in \mathrm{Num}_0(\bar{X})$. But this is clear for both things that are pulled back and things that are E. So we are done. □

Theorem (Elimination of indeterminacy)**.** Let X be a sooth projective surface, Y a projective variety, and $\varphi : X \to Y$ a rational map. Then there exists a smooth projective surface X' and a commutative diagram



where $p : X' \to X$ is a composition of blow ups, and in particular is birational.

Proof. We may assume $Y = \mathbb{P}^n$, and $X \dashrightarrow \mathbb{P}^n$ is non-degenerate. Then φ is induced by a linear system $|V| \subseteq H^0(X, \mathcal{L})$. We first show that we may assume the base locus has codimension > 1.

If not, every element of $|V|$ is of the form $C + D'$ for some fixed C. Let $|V'|$ be the set of all such D'. Then $|V|$ and $|V'|$ give the same rational map. Indeed, if V has basis $f_0, \ldots, f_n$, and g is the function defining C, then the two maps are, respectively,

$$[f_0 : \cdots : f_n] \text{ and } [f_0/g : \cdots : f_n/g],$$

which define the same rational maps. By repeating this process, replacing V with V', we may assume the base locus has codimension > 1.

We now want to show that by blowing up, we may remove all points from the base locus. We pick $x \in X$ in the base locus of $|D|$. Blow it up to get $X_1 = \mathrm{Bl}_x X \to X$. Then

$$\pi^*|D| = |D_1| + mE,$$

where $m > 0$, and $|D_1|$ is a linear system which induces the map $\varphi \circ \pi_1$. If $|D_1|$ has no basepoints, then we are done. If not, we iterate this procedure. The procedure stops, because

$$0 \leq D_1^2 = (\pi_1^* D^2 + m^2 E^2) < D^2. \qquad \square$$

Theorem. Let $g : Z \to X$ be a birational morphism of surfaces. Then g factors as $Z \overset{g'}{\to} X' \overset{p}{\to} X$, where $p : X' \to X$ is a composition of blow ups, and g' is an isomorphism.

Proof. Apply elimination of indeterminacy to the rational inverse to g to obtain $p : X' \to X$. There is then a lift of g' to X' by the universal property, and these are inverses to each otehr. $\square$

Theorem (Castelnuovo's contractibility criterion)**.** Let X be a smooth projective surface over $K = \bar{K}$. If there is a curve $C \subseteq X$ such that $C \cong \mathbb{P}^1$ and $C^2 = -1$, then there exists $f : X \to Y$ that exhibits X as a blow up of Y at a point with exceptional curve C.

Corollary. A smooth projective surface is relatively minimal if and only if it does not contain a (-1) curve. $\square$

Proof. The idea is to produce a suitable linear system $|L|$, giving a map $f : X \to \mathbb{P}^n$, and then take Y to be the image. Note that $f(C) = *$ is the same as requiring

$$\mathcal{O}_X(L)|_C = \mathcal{O}_C.$$

After finding a linear system that satisfies this, we will do some work to show that f is an isomorphism outside of C and has smooth image.

Let H be a very ample divisor on X. By Serre's theorem, we may assume $H^1(X, \mathcal{O}_X(H)) = 0$. Let

$$H \cdot C = K > 0.$$

Consider divisors of the form $H + iC$. We can compute

$$\deg_C (H + iC)|_C = (H + iC) \cdot C = K - i.$$

Thus, we know $\deg_C(H+KC)|_C = 0$, and hence

$$\mathcal{O}_X(H+KC)|_C \cong \mathcal{O}_C.$$

We claim that $\mathcal{O}_X(H+KC)$ works. To check this, we pick a fairly explicit basis of $H^0(H+KC)$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(H+(i-1)C) \to \mathcal{O}_X(H+iC) \to \mathcal{O}_C(H+iC) \to 0,$$

inducing a long exact sequence

$$\begin{aligned} 0 \to H^0(H+(i-1)C) \to H^0(H+iC) \to H^0(C,(H+iC)|_C) \\ \to H^1(H+(i-1)C) \to H^1(H+iC) \to H^1(C,(H+iC)|_C) \to \cdots \end{aligned}$$

We know $\mathcal{O}_X(H+iC)|_C = \mathcal{O}_{\mathbb{P}^1}(K-i)$. So in the range $i=0,\ldots,K$, we have $H^1(C,(H+iC)|_C) = 0$. Thus, by induction, we find that

$$H^1(H+iC) = 0 \text{ for } i = 0,\ldots,K.$$

As a consequence of this, we know that for $i=1,\ldots,K$, we have a short exact sequence

$$H^0(H+(i-1)C) \hookrightarrow H^0(H+iC) \twoheadrightarrow H^0(C,(H+iC)|_C) = H^0(\mathcal{O}_{\mathbb{P}^1}(K-i)).$$

Thus, $H^0(H+iC)$ is spanned by the image of $H^0(H+(i-1)C)$ plus a lift of a basis of $H^0(\mathcal{O}_{\mathbb{P}^1}(K-i))$.

For each $i>0$, pick a lift $y_0^{(i)},\ldots,y_{K-i}^{(i)}$ of a basis of $H^0(\mathcal{O}_{\mathbb{P}^1}(K-i))$ to $H^0(H+iC)$, and take the image in $H^0(H+KC)$. Note that in local coordinates, if C is cut out by a function g, then the image in $H^0(H+KC)$ is given by

$$g^{K-i}y_0^{(i)},\ldots,g^{K-i}y_{K-i}^{(i)}.$$

For $i=0$, we pick a basis of $H^0(H)$, and then map it down to $H^0(H+KC)$. Taking the union over all i of these elements, we obtain a basis of $H^0(H+KC)$. Let f be the induced map to $\mathbb{P}^r$.

For concreteness, list the basis vectors as $x_1,\ldots,x_r$, where x_r is a lift of $1 \in \mathcal{O}_{\mathbb{P}^1}$ to $H^0(H+KC)$, and x_{r-1},x_{r-2} are $gy_0^{(K-1)}, gy_1^{(K-1)}$.

First observe that $x_1,\ldots,x_{r-1}$ vanish at C. So

$$f(C) = [0:\cdots:0:1] \equiv p,$$

and C is indeed contracted to a point.

Outside of C, the function g is invertible, so just the image of $H^0(H)$ is enough to separate points and tangent vectors. So f is an isomorphism outside of C.

All the other basis elements of $H+KC$ are needed to make sure the image is smooth, and we only have to check this at the point p. This is done by showing that $\dim \mathfrak{m}_{Y,p}/\mathfrak{m}_{Y,p}^2 \leq 2$, which requires some "infinitesimal analysis" we will not perform. □

3 Projective varieties

3.1 The intersection product

Lemma. Assume $D \equiv 0$. Then for all $D_2, \ldots, D_{\dim X}$, we have

$$D \cdot D_2 \cdot \ldots \cdot D_{\dim X} = 0.$$

Proof. We induct on n. As usual, we can assume that the D_i are very ample. Then

$$D \cdot D_2 \cdot \cdots D_{\dim X} = D|_{D_2} \cdot D_3|_{D_2} \cdot \cdots \cdot D_{\dim X}|_{D_2}.$$

Now if $D \equiv 0$ on X, then $D|_{D_2} \equiv 0$ on D_2. So we are done. □

Theorem (Severi)**.** Let X be a projective variety over an algebraically closed field. Then $N^1(X)$ is a finitely-generated torsion free abelian group, hence of the form $\mathbb{Z}^n$. □

Theorem (Asymptotic Riemann–Roch)**.** Let X be a projective normal variety over $K = \bar{K}$. Let D be a Cartier divisor, and E a Weil divisor on X. Then $\chi(X, \mathcal{O}_X(mD + E))$ is a numerical polynomial in m (i.e. a polynomial with rational coefficients that only takes integral values) of degree at most $n = \dim X$, and

$$\chi(X, \mathcal{O}_X(mD + E)) = \frac{D^n}{n!} m^n + \text{lower order terms}.$$

Proof. By induction on $\dim X$, we can assume the theorem holds for normal projective varieties of dimension $< n$. We fix H on X very ample such that $H + D$ is very ample. Let $H' \in |H|$ and $G \in |H + D|$ be sufficiently general. We then have short exact sequences

$$0 \to \mathcal{O}_X(mD + E) \to \mathcal{O}_X(mD + E + H) \to \mathcal{O}_{H'}((mD + E + H)|_{H'}) \to 0$$

$$0 \to \mathcal{O}_X((m-1)D + E) \to \mathcal{O}_X(mD + E + H) \to \mathcal{O}_G((mD + E + H)|_G) \to 0.$$

Note that the middle term appears in both equations. So we find that

$$\begin{aligned}\chi(X, \mathcal{O}_X(mD + E)) &+ \chi(H', \mathcal{O}_{H'}((mD + E + H)|_H)) \\ &= \chi(X, \mathcal{O}_X((m-1)D + E)) + \chi(H', \mathcal{O}_G((mD + E + H)|_G))\end{aligned}$$

Rearranging, we get

$$\begin{aligned}\chi(\mathcal{O}_X(mD - E)) &- \chi(\mathcal{O}_X((m-1)D - E)) \\ &= \chi(G, \mathcal{O}_G(mD + E + H)) - \chi(H', \mathcal{O}_{H'}(mD + E + H)).\end{aligned}$$

By induction (see proposition below) the right-hand side is a numerical polynomial of degree at most $n - 1$ with leading term

$$\frac{D^{n-1} \cdot G - D^{n-1} \cdot H}{(n-1)!} m^{n-1} + \text{lower order terms},$$

since $D^{n-1} \cdot G$ is just the $(n-1)$ self-intersection of D along G. But $G - H = D$. So the LHS is

$$\frac{D^n}{(n-1)!} m^{n-1} + \text{lower order terms},$$

so we are done. □

Proposition. Let X be a normal projective variety, and $|H|$ a very ample linear system. Then for a general element $G \in |H|$, G is a normal projective variety. □

Proposition. Let X be a normal projective variety.

(i) If H is a very ample Cartier divisor, then

$$h^0(X, mH) = \frac{H^n}{n!} m^n + \text{lower order terms for } m \gg 0.$$

(ii) If D is any Cartier divisor, then there is some $C \in \mathbb{R}_{>0}$ such that

$$h^0(mD) \leq C \cdot m^n \text{ for } m \gg 0.$$

Proof.

(i) By Serre's theorem, $H^i(\mathcal{O}_x(mH)) = 0$ for $i > 0$ and $m \gg 0$. So we apply asymptotic Riemann Roch.

(ii) There exists a very ample Cartier divisor H' on X such that $H' + D$ is also very ample. Then

$$h^0(mD) \leq h^0(m(H' + D)).$$ □

3.2 Ample divisors

Lemma. Let X, Y be projective schemes. If $f : X \to Y$ is a finite morphism of schemes, and D is an ample Cartier divisor on Y, then so is f^*D.

Proof. Let $\mathcal{F}$ be a coherent sheaf on X. Since f is finite, we have $R^i f_* \mathcal{F} = 0$ for all $i > 0$. Then

$$H^i(\mathcal{F} \otimes f^* \mathcal{O}_X(mD)) = H^i(f_* \mathcal{F} \otimes \mathcal{O}_Y(mD)) = 0$$

for all $i > 0$ and $m \gg 0$. So by Serre's theorem, we know f^*D is ample. □

Proposition. Let X be a proper scheme, and $\mathcal{L}$ an invertible sheaf. Then $\mathcal{L}$ is ample iff $\mathcal{L}|_{X_{\mathrm{red}}}$ is ample.

Proof.

(⇒) If $\mathcal{L}$ induces a closed embedding of X, then map given by $\mathcal{L}|_{X_{\mathrm{red}}}$ is given by the composition with the closed embedding $X_{\mathrm{red}} \hookrightarrow X$.

(⇐) Let $\mathcal{J} \subseteq \mathcal{O}_X$ be the nilradical. By Noetherianness, there exists n such that $\mathcal{J}^n = 0$.

Fix $\mathcal{F}$ a coherent sheaf. We can filter $\mathcal{F}$ using

$$\mathcal{F} \supseteq \mathcal{J}\mathcal{F} \supseteq \mathcal{J}^2\mathcal{F} \supseteq \cdots \supseteq \mathcal{J}^{n-1}\mathcal{F} \supseteq \mathcal{J}^n\mathcal{F} = 0.$$

For each j, we have a short exact sequence

$$0 \to \mathcal{J}^{j+1}\mathcal{F} \to \mathcal{J}^j\mathcal{F} \to \mathcal{G}_j \to 0.$$

This $\mathcal{G}_i$ is really a sheaf on the reduced structure, since $\mathcal{J}$ acts trivially. Thus $H^i(\mathcal{G}_j \otimes \mathcal{L}^m)$ for $j > 0$ and large m. Thus inducting on $j \geq 0$, we find that for $i > 0$ and $m \gg 0$, we have

$$H^i(\mathcal{J}^j\mathcal{F} \otimes \mathcal{L}^m) = 0.$$ □

Theorem (Nakai's criterion)**.** Let X be a projective variety. Let D be a Cartier divisor on X. Then D is ample iff for all $V \subseteq X$ integral proper subvariety (proper means proper scheme, not proper subset), we have

$$(D|_V)^{\dim V} = D^{\dim V}[V] > 0.$$

Corollary. Let X be a projective variety. Then ampleness is a numerical condition, i.e. for any Cartier divisors D_1, D_2, if $D_1 \equiv D_2$, then D_1 is ample iff D_2 is ample.

Corollary. Let X, Y be projective variety. If $f : X \to Y$ is a surjective finite morphism of schemes, and D is a Cartier divisor on Y. Then D is ample iff f^*D is ample.

Proof. It remains to prove $\Leftarrow$. If f is finite and surjective, then for all $V \subseteq Y$, there exists $V' \subseteq f^{-1}(V) \subseteq X$ such that $f|_{V'} : V' \to V$ is a finite surjective morphism. Then we have

$$(f^*D)^{\dim V'}[V'] = \deg f|_{V'} D^{\dim V}[V],$$

which is clear since we only have to prove this for very ample D. □

Corollary. If X is a projective variety, D a Cartier divisor and $\mathcal{O}_X(D)$ globally generated, and

$$\Phi : X \to \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*)$$

the induced map. Then D is ample iff Φ is finite.

Proof.

($\Leftarrow$) If $X \to \Phi(X)$ is finite, then $D = \Phi^*\mathcal{O}(1)$. So this follows from the previous corollary.

($\Rightarrow$) If Φ is not finite, then there exists $C \subseteq X$ such that $\Phi(C)$ is a point. Then $D \cdot [C] = \Phi^*\mathcal{O}(1) \cdot [C] = 0$, by the push-pull formula. So by Nakai's criterion, D is not ample. □

Proof of Nakai's criterion.

($\Rightarrow$) If D is ample, then mD is very ample for some m. Then by multilinearity, we may assume D is very ample. So we have a closed embedding

$$\Phi : X \to \mathbb{P}(H^0(D)^*).$$

If $V \subseteq X$ is a closed integral subvariety, then $D^{\dim V} \cdot [V] = (D|_V)^{\dim V}$. But this is just $\deg_{\Phi(V)} \mathcal{O}(1) > 0$.

($\Leftarrow$) We proceed by induction on $\dim X$, noting that $\dim X = 1$ is trivial. By induction, for any proper subvariety V, we know that $D|_V$ is ample.

The key of the proof is to show that $\mathcal{O}_X(mD)$ is globally generated for large m. If so, the induced map $X \to \mathbb{P}(|mD|)$ cannot contract any curve C, or else $mD \cdot C = 0$. So this is a finite map, and mD is the pullback of the ample divisor $\mathcal{O}_{\mathbb{P}(|mD|)}(1)$ along a finite map, hence is ample.

We first reduce to the case where D is effective. As usual, write $D \sim H_1 - H_2$ with H_i very ample effective divisors. We have exact sequences

$$0 \to \mathcal{O}_X(mD - H_1) \to \mathcal{O}_X(mD) \to \mathcal{O}_{H_1}(mD) \to 0$$

$$0 \to \mathcal{O}_X(mD - H_1) \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_{H_2}((m-1)D) \to 0.$$

We know $D|_{H_i}$ is ample by induction. So the long exact sequences implies that for all $m \gg 0$ and $j \geq 2$, we have

$$H^j(mD) \cong H^j(mD - H_1) = H^j((m-1)D).$$

So we know that

$$\chi(mD) = h^0(mD) - h^1(mD) + \text{constant}$$

for all $m \gg 0$. On the other hand, since X is an integral subvariety of itself, $D^n > 0$, and so asymptotic Riemann–Roch tells us $h^0(mD) > 0$ for all $m \gg 0$. Since D is ample iff mD is ample, we can assume D is effective.

To show that mD is globally generated, we observe that it suffices to show that this is true in a neighbourhood of D, since outside of D, the sheaf is automatically globally generated by using the tautological section that vanishes at D with multiplicity m.

Moreover, we know $mD|_D$ is very ample, and in particular globally generated for large m by induction (the previous proposition allows us to pass to $D|_{red}$ if necessary). Thus, it suffices to show that

$$H^0(\mathcal{O}_X(mD)) \to H^0(\mathcal{O}_D(mD))$$

is surjective.

To show this, we use the short exact sequence

$$0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_D(mD) \to 0.$$

For $i > 0$ and large m, we know $H^i(\mathcal{O}_D(mD)) = 0$. So we have surjections

$$H^1((m-1)D) \twoheadrightarrow H^1(mD)$$

for m large enough. But these are finite-dimensional vector spaces. So for m sufficiently large, this map is an isomorphism. Then $H^0(\mathcal{O}_X(mD)) \to H^0(\mathcal{O}_D(mD))$ is a surjection by exactness. □

Proposition. Let $D \in \mathrm{CaDiv}_{\mathbb{Q}}(X)$ Then the following are equivalent:

(i) cD is an ample integral divisor for some $c \in \mathbb{N}_{>0}$.

(ii) $D = \sum c_i D_i$, where $c_i \in \mathbb{Q}_{>0}$ and D_i are ample Cartier divisors.

(iii) D satisfies Nakai's criterion. That is, $D^{\dim V}[V] > 0$ for all integral subvarieties $V \subseteq X$.

Proof. It is easy to see that (i) and (ii) are equivalent. It is also easy to see that (i) and (iii) are equivalent. □

Lemma. A positive linear combination of ample divisors is ample.

Proof. Let H_1, H_2 be ample. Then for $\lambda_1, \lambda_2 > 0$, we have

$$(\lambda_1 H_1 + \lambda_2 H_2)^{\dim V}[V] = \left(\sum \binom{\dim V}{p} \lambda_1^p \lambda_2^{\dim V - p} \, H_1^p \cdot H_2^{\dim V - p}\right)[V]$$

Since any restriction of an ample divisor to an integral subscheme is ample, and multiplying with H is the same as restricting to a hyperplane cuts, we know all the terms are positive. □

Proposition. Ampleness is an open condition. That is, if D is ample and $E_1, \ldots E_r$ are Cartier divisors, then for all $|\varepsilon_i| \ll 1$, the divisor $D + \varepsilon_i E_i$ is still ample.

Proof. By induction, it suffices to check it in the case $n = 1$. Take $m \in \mathbb{N}$ such that $mD \pm E_1$ is still ample. This is the same as saying $D \pm \frac{1}{m} E_1$ is still ample.

Then for $|\varepsilon_1| < \frac{1}{m}$, we can write

$$D + \varepsilon E_1 = (1 - q)D + q\left(D + \frac{1}{m} E_1\right)$$

for some $q < 1$. □

Proposition. Being ample is a numerical property over $\mathbb{R}$, i.e. if $D_1, D_2 \in \mathrm{CaDiv}_{\mathbb{R}}(X)$ are such that $D_1 \equiv D_2$, then D_1 is ample iff D_2 is ample.

Proof. We already know that this is true over $\mathbb{Q}$ by Nakai's criterion. Then for real coefficients, we want to show that if D is ample, E is numerically trivial and $t \in \mathbb{R}$, then $D + tE$ is ample. To do so, pick $t_1 < t < t_2$ with $t_i \in \mathbb{Q}$, and then find $\lambda, \mu > 0$ such that

$$\lambda(D_1 + t_1 E) + \mu(D_1 + t_2 E) = D_1 + tE.$$

Then we are done by checking Nakai's criterion. □

Proposition. Let H be an ample $\mathbb{R}$-divisor. Then for all $\mathbb{R}$-divisors $E_1, \ldots, E_r$, for all $\|\varepsilon_i\| \leq 1$, the divisor $H + \sum \varepsilon_i E_i$ is still ample. □

3.3 Nef divisors

Proposition.

(i) D is nef iff $D|_{X_{\mathrm{red}}}$ is nef.

(ii) D is nef iff $D|_{X_i}$ is nef for all irreducible components X_i.

(iii) If $V \subseteq X$ is a proper subscheme, and D is nef, then $D|_V$ is nef.

(iv) If $f : X \to Y$ is a finite morphism of proper schemes, and D is nef on Y, then $f^* D$ is nef on X. The converse is true if f is surjective. □

Theorem (Kleinmann's criterion)**.** Let X be a proper scheme, and D an $\mathbb{R}$-Cartier divisor. Then D is nef iff $D^{\dim V}[V] \geq 0$ for all proper irreducible subvarieties.

Corollary. Let X be a projective scheme, and D be a nef $\mathbb{R}$-divisor on X, and H be a Cartier divisor on X.

(i) If H is ample, then $D + \varepsilon H$ is also ample for all $\varepsilon > 0$.

(ii) If $D + \varepsilon H$ is ample for all $0 < \varepsilon \ll 1$, then D is nef.

Proof.

(i) We may assume H is very ample. By Nakai's criterion this is equivalent to requiring

$$(D + \varepsilon H)^{\dim V} \cdot [V] = \left(\sum \binom{\dim V}{p} \varepsilon^p D^{\dim V - p} H^p \right) [V] > 0.$$

Since any restriction of a nef divisor to any integral subscheme is also nef, and multiplying with H is the same as restricting to a hyperplane cuts, we know the terms that involve D are non-negative. The H^p term is positive. So we are done.

(ii) We know $(D + \varepsilon H) \cdot C > 0$ for all positive ε sufficiently small. Taking $\varepsilon \to 0$, we know $D \cdot C \geq 0$. □

Corollary. $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$ and $\mathrm{int}(\mathrm{Nef}(X)) = \mathrm{Amp}(X)$.

Proof. We know $\mathrm{Amp}(X) \subseteq \mathrm{Nef}(X)$ and $\mathrm{Amp}(X)$ is open. So this implies $\mathrm{Amp}(X) \subseteq \mathrm{int}(\mathrm{Nef}(X))$, and thus $\overline{\mathrm{Amp}(X)} \subseteq \mathrm{Nef}(X)$.

Conversely, if $D \in \mathrm{int}(\mathrm{Nef}(X))$, we fix H ample. Then $D - tH \in \mathrm{Nef}(X)$ for small t, by definition of interior. Then $D = (D - tH) + tH$ is ample. So $\mathrm{Amp}(X) \supseteq \mathrm{int}(\mathrm{Nef}(X))$. □

Proof of Kleinmann's criterion. We may assume that X is an integral projective scheme. The $\Leftarrow$ direction is immediate. To prove the other direction, since the criterion is a closed condition, we may assume $D \in \mathrm{Div}_{\mathbb{Q}}(X)$. Moreover, by induction, we may assume that $D^{\dim V}[V] \geq 0$ for all V strictly contained in X, and we have to show that $D^{\dim X} \geq 0$. Suppose not, and $D^{\dim X} < 0$.

Fix a very ample Cartier divisor H, and consider the polynomial

$$P(t) = (D + tH)^{\dim X} = D^{\dim X} + \sum_{i=1}^{\dim X - 1} t^i \binom{\dim X}{i} H^i D^{\dim X - i} + t^{\dim X} H^{\dim X}.$$

The first term is negative; the last term is positive; and the other terms are non-negative by induction since H is very ample.

Then on $\mathbb{R}_{>0}$, this polynomial is increasing. So there exists a unique t such that $P(t) = 0$. Let $\bar{t}$ be the root. Then $P(\bar{t}) = 0$. We can also write

$$P(t) = (D + tH) \cdot (D + tH)^{\dim X - 1} = R(t) + tQ(t),$$

where

$$R(t) = D \cdot (D + tH)^{\dim X - 1}, \quad Q(t) = H \cdot (D + tH)^{\dim X - 1}.$$

We shall prove that $R(\bar{t}) \geq 0$ and $Q(\bar{t}) > 0$, which is a contradiction.

We first look at $Q(t)$, which is

$$Q(t) = \sum_{i=0}^{\dim X - 1} t^i \binom{\dim X - 1}{i} H^{i+1} D^{\dim X - i},$$

which, as we argued, is a sum of non-negative terms and a positive term.

To understand $R(t)$, we look at

$$R(t) = D \cdot (D + tH)^{\dim X - 1}.$$

Note that so far, we haven't used the assumption that D is nef. If $t > \bar{t}$, then $(D + tH)^{\dim X} > 0$, and $(D + tH)^{\dim V}[V] > 0$ for a proper integral subvariety, by induction (and binomial expansion). Then by Nakai's criterion, $D + tH$ is ample. So the intersection $(D + tH)^{\dim X - 1}$ is essentially a curve. So we are done by definition of nef. Then take the limit $t \to \bar{t}$. □

Proposition. $\mathrm{Nef}(X) = \mathrm{NE}(X)^\vee$. □

Proposition. $\overline{NE(X)} = \mathrm{Nef}(X)^\vee$. □

Theorem (Kleinmann's criterion)**.** If X is a projective scheme and $D \subseteq \mathrm{CaDiv}_\mathbb{R}(X)$. Then the following are equivalent:

(i) D is ample

(ii) $D|_{\overline{NE(X)}} > 0$, i.e. $D \cdot \gamma > 0$ for all $\gamma \in \overline{NE(X)}$.

(iii) $\mathbb{S}_1 \cap \overline{NE(X)} \subseteq \mathbb{S}_1 \cap D_{>0}$, where $\mathbb{S}_1 \subseteq N_1(X)_\mathbb{R}$ is the unit sphere under some choice of norm.

Proof.

- (1) ⇒ (2): Trivial.
- (2) ⇒ (1): If $D|_{\overline{NE(X)}} > 0$, then $D \in \mathrm{int}(\mathrm{Nef}(X))$.
- (2) ⇔ (3): Similar. □

Proposition. Let X be a projective scheme, and $D, H \in N^1_\mathbb{R}(X)$. Assume that H is ample. Then D is ample iff there exists $\varepsilon > 0$ such that

$$\frac{\mathrm{D} \cdot C}{H \cdot C} \geq \varepsilon.$$

Proof. The statement in the lemma is the same as $(D - \varepsilon H) \cdot C \geq 0$. □

Theorem (Cone theorem)**.** Let X be a smooth projective variety over $\mathbb{C}$. Then there exists rational curves $\{C_i\}_{i \in I}$ such that

$$\overline{\mathrm{NE}(X)} = \overline{\mathrm{NE}}_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+[C_i]$$

where $\overline{\mathrm{NE}}_{K_X \geq 0} = \{\gamma \in \overline{\mathrm{NE}(X)} : K_X \cdot \gamma \geq 0\}$. Further, we need at most countably many C_i's, and the accumulation points are all at $K_X^\perp$.

3.4 Kodaira dimension

Theorem (Iitaka)**.** Let X be a normal projective variety and $\mathcal{L}$ a line bundle on X. Suppose there is an m such that $|\mathcal{L}^{\otimes m}| \neq 0$. Then there exists X_∞, Y_∞, a map $\psi_\infty : X_\infty \to Y_\infty$ and a birational map $U_\infty : X_\infty \dashrightarrow X$ such that for $K \gg 0$ such that $|\mathcal{L}^{\otimes K}| \neq 0$, we have a commutative diagram

$$\begin{array}{ccc} X & \overset{\varphi_{|\mathcal{L}^{\otimes k}|}}{\dashrightarrow} & \mathrm{Im}(\varphi_{|\mathcal{L}^{\otimes K}|}) \\ {\scriptstyle U_\infty}\uparrow & & \uparrow \\ X_\infty & \xrightarrow{\psi_\infty} & Y_\infty \end{array}$$

where the right-hand map is also birational.