Part III — Positivity in Algebraic Geometry
Theorems

Based on lectures by S. Svaldi
Notes taken by Dexter Chua

Lent 2018

This class aims at giving an introduction to the theory of divisors, linear systems and their positivity properties on projective algebraic varieties.

The first part of the class will be dedicated to introducing the basic notions and results regarding these objects and special attention will be devoted to discussing examples in the case of curves and surfaces.

In the second part, the course will cover classical results from the theory of divisors and linear systems and their applications to the study of the geometry of algebraic varieties.

If time allows and based on the interests of the participants, there are a number of more advanced topics that could possibly be covered: Reider's Theorem for surfaces, geometry of linear systems on higher dimensional varieties, multiplier ideal sheaves and invariance of plurigenera, higher dimensional birational geometry.

Pre-requisites

The minimum requirement for those students wishing to enroll in this class is their knowledge of basic concepts from the Algebraic Geometry Part 3 course, i.e. roughly Chapters 2 and 3 of Hartshorne's Algebraic Geometry.

Familiarity with the basic concepts of the geometry of algebraic varieties of dimension 1 and 2 — e.g. as covered in the preliminary sections of Chapters 4 and 5 of Hartshorne's Algebraic Geometry — would be useful but will not be assumed — besides what was already covered in the Michaelmas lectures.

Students should have also some familiarity with concepts covered in the Algebraic Topology Part 3 course such as cohomology, duality and characteristic classes.

Contents

1 Divisors

1.1 Projective embeddings

Theorem. Let A be any ring, and X a scheme over A.

(i) If $\varphi : X \to \mathbb{P}^n$ is a morphism over A, then $\varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$ is an invertible sheaf on X, generated by the sections $\varphi^* x_0, \ldots, \varphi^* x_n \in H^0(X, \varphi^*\mathcal{O}_{\mathbb{P}^n}(1))$.

(ii) If $\mathcal{L}$ is an invertible sheaf on X, and if $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ which generate $\mathcal{L}$, then there exists a unique morphism $\varphi : X \to \mathbb{P}^n$ such that $\varphi^*\mathcal{O}(1) \cong \mathcal{L}$ and $\varphi^* x_i = s_i$.

Proposition. Let $K = \bar{K}$, and X a projective variety over K. Let $\mathcal{L}$ be an invertible sheaf on X, and $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ generating sections. Write $V = \langle s_0, \ldots, s_n \rangle$ for the linear span. Then the associated map $\varphi : X \to \mathbb{P}^n$ is a closed embedding iff

(i) For every distinct closed points $p \neq q \in X$, there exists $s_{p,q} \in V$ such that $s_{p,q} \in \mathfrak{m}_p \mathcal{L}_p$ but $s_{p,q} \notin \mathfrak{m}_q \mathcal{L}_q$.

(ii) For every closed point $p \in X$, the set $\{s \in V \mid s \in \mathfrak{m}_p \mathcal{L}_p\}$ spans the vector space $\mathfrak{m}_p \mathcal{L}_p / \mathfrak{m}_p^2 \mathcal{L}_p$.

Lemma. Let $f : A \to B$ be a local morphism of local rings such that

- $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism;
- $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective; and
- B is a finitely-generated A-module.

Then f is surjective. □

Theorem (Serre)**.** Let X be a projective scheme over a Noetherian ring A, $\mathcal{L}$ be a very ample invertible sheaf, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then there exists a positive integer $n_0 = n_0(\mathcal{F}, \mathcal{L})$ such that for all $n \geq n_0$, the twist $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. □

Theorem (Serre)**.** Let X be a scheme of finite type over a Noetherian ring A, and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample iff there exists $m > 0$ such that $\mathcal{L}^m$ is very ample.

Proposition. Let $\mathcal{L}$ be a sheaf over X (which is itself a projective variety over K). Then the following are equivalent:

(i) $\mathcal{L}$ is ample.

(ii) $\mathcal{L}^m$ is ample for all $m > 0$.

(iii) $\mathcal{L}^m$ is ample for some $m > 0$. □

Theorem (Serre)**.** Let X be a projective scheme over a Noetherian ring A, and $\mathcal{L}$ is very ample on X. Let $\mathcal{F}$ be a coherent sheaf. Then

(i) For all $i \geq 0$ and $n \in \mathbb{N}$, $H^i(\mathcal{F} \otimes \mathcal{L}^n)$ is a finitely-generated A-module.

(ii) There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $H^i(\mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $i > 0$. □

Theorem. Let X be a proper scheme over a Noetherian ring A, and $\mathcal{L}$ an invertible sheaf. Then the following are equivalent:

(i) $\mathcal{L}$ is ample.

(ii) For all coherent $\mathcal{F}$ on X, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $H^i(\mathcal{F} \otimes \mathcal{L}^n) = 0$.

1.2 Weil divisors

Theorem (Hartog's lemma)**.** Let X be normal, and $f \in \mathcal{O}(X \setminus V)$ for some $V \geq 2$. Then $f \in \mathcal{O}_X$. Thus, $\operatorname{div}(f) = 0$ implies $f \in \mathcal{O}_X^\times$.

1.3 Cartier divisors

Proposition. If X is normal, then

$$\operatorname{div} : \{\text{rational sections of } \mathcal{L}\} \to \operatorname{WDiv}(X).$$

is well-defined, and two sections have the same image iff they differ by an element of $\mathcal{O}_X^*$.

Corollary. If X is normal and proper, then there is a map

$$\operatorname{div}\{\text{rational sections of } \mathcal{L}\}/K^* \to \operatorname{WDiv}(X).$$

Proposition. $\mathcal{O}_X(D)$ is a rank 1 quasicoherent $\mathcal{O}_X$-module. □.

Proposition. If D is locally principal at every point x, then $\mathcal{O}_X(D)$ is an invertible sheaf.

Proposition. If D_1, D_2 are Cartier divisors, then

(i) $\mathcal{O}_X(D_1 + D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$.

(ii) $\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^\vee$.

(iii) If $f \in K(X)$, then $\mathcal{O}_X(\operatorname{div}(f)) \cong \mathcal{O}_X$. □

Proposition. Let X be a Noetherian, normal, integral scheme. Assume that X is *factorial*, i.e. every local ring $\mathcal{O}_{X,x}$ is a UFD. Then any Weil divisor is Cartier.

Theorem. Let X be normal and $\mathcal{L}$ an invertible sheaf, s a rational section of $\mathcal{L}$. Then $\mathcal{O}_X(\operatorname{div}(s))$ is invertible, and there is an isomorphism

$$\mathcal{O}_X(\operatorname{div}(s)) \to \mathcal{L}.$$

Moreover, sending $\mathcal{L}$ to $\operatorname{div} s$ gives a map

$$\operatorname{Pic}(X) \to \operatorname{Cl}(X),$$

which is an isomorphism if X is factorial (and Noetherian and integral).

1.4 Computations of class groups

Proposition. Let X be an integral scheme, regular in codimension 1. If $Z \subseteq X$ is an integral closed subscheme of codimension 1, then we have an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus Z) \to 0,$$

where $n \in \mathbb{Z}$ is mapped to $[nZ]$.

Proposition. If $Z \subseteq X$ has codimension ≥ 2, then $\mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus Z)$ is an isomorphism.

Proposition. If A is a Noetherian ring, regular in codimension 1, then A is a UFD iff A is normal and $\mathrm{Cl}(\operatorname{Spec} A) = 0$

Proposition. Let X be Noetherian and regular in codimension one. Then

$$\mathrm{Cl}(X) = \mathrm{Cl}(X \times \mathbb{A}^1).$$

1.5 Linear systems

Proposition. Let X be a smooth projective variety over an algebraically closed field. Let D_0 be a divisor on X.

(i) For all $s \in H^0(X, \mathcal{O}_X(D))$, $\mathrm{div}(s)$ is an effective divisor linearly equivalent to D.

(ii) If $D \sim D_0$ and $D \geq 0$, then there is $s \in H^0(\mathcal{O}_X(D_0))$ such that $\mathrm{div}(s) = D$

(iii) If $s, s' \in H^0(\mathcal{O}_X(D_0))$ and $\mathrm{div}(s) = \mathrm{div}(s')$, then $s' = \lambda s$ for some $\lambda \in K^*$.

Theorem (Riemann–Roch theorem)**.** If C is a smooth projective curve, then

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g(C). \qquad \square$$

Proposition. Let D be a Cartier divisor on a projective normal scheme .Then $D \sim H_1 - H_2$ for some very ample divisors H_i. We can in fact take H_i to be effective, and if X is smooth, then we can take H_i to be smooth and intersecting transversely. $\square$

Theorem (Bertini)**.** Let X be a smooth projective variety over an algebraically closed field K, and D a very ample divisor. Then there exists a Zariski open set $U \subseteq |D|$ such that for all $H \in U$, H is smooth on X and if $H_1 \neq H_2$, then H_1 and H_2 intersect transversely. $\square$

2 Surfaces

2.1 The intersection product

Proposition.

(i) The product $D_1 \cdot D_2$ depends only on the classes of D_1, D_2 in $\mathrm{Pic}(X)$.

(ii) $D_1 \cdot D_2 = D_2 \cdot D_1$.

(iii) $D_1 \cdot D_2 = |D_1 \cap D_2|$ if D_1 and D_2 are curves intersecting transversely.

(iv) The intersection product is bilinear.

Theorem (Riemann–Roch for surfaces)**.** Let $D \in \mathrm{Div}(X)$. Then

$$\chi(X, \mathcal{O}_X(D)) = \frac{D \cdot (D - K_X)}{2} + \chi(\mathcal{O}_X),$$

where K_X is the *canonical divisor*.

Theorem (Adjunction formula)**.** Let X be a smooth surface, and $C \subseteq X$ a smooth curve. Then

$$(\mathcal{O}_X(K_X) \otimes \mathcal{O}_X(C))|_C \cong \mathcal{O}_C(K_C).$$

Theorem (Hodge index theorem)**.** Let X be a projective surface, and H be a (very) ample divisor on X. Let D be a divisor on X such that $D \cdot H = 0$ but $D \not\equiv 0$. Then $D^2 < 0$.

2.2 Blow ups

Lemma.

$$\pi^* C = \tilde{C} + mE,$$

where m is the multiplicity of C at p.

Proposition. Let X be a smooth projective surface, and $x \in X$. Let $\bar{X} = \mathrm{Bl}_x X \overset{\pi}{\to} X$. Then

(i) $\pi^* \mathrm{Pic}(X) \oplus \mathbb{Z}[E] = \mathrm{Pic}(\bar{X})$

(ii) $\pi^* D \cdot \pi^* F = D \cdot F, \quad \pi^* D \cdot E = 0, \quad E^2 = -1.$

(iii) $K_{\bar{X}} = \pi^*(K_X) + E$.

(iv) π^* is defined on $\mathrm{NS}(X)$. Thus,

$$\mathrm{NS}(\bar{X}) = \mathrm{NS}(X) \oplus \mathbb{Z}[E].$$

Theorem (Elimination of indeterminacy)**.** Let X be a sooth projective surface, Y a projective variety, and $\varphi : X \to Y$ a rational map. Then there exists a smooth projective surface X' and a commutative diagram

$$\begin{tikzcd} & X' \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\varphi"] & & Y \end{tikzcd}$$

where $p : X' \to X$ is a composition of blow ups, and in particular is birational.

Theorem. Let $g : Z \to X$ be a birational morphism of surfaces. Then g factors as $Z \xrightarrow{g'} X' \xrightarrow{p} X$, where $p : X' \to X$ is a composition of blow ups, and g' is an isomorphism.

Theorem (Castelnuovo's contractibility criterion)**.** Let X be a smooth projective surface over $K = \bar{K}$. If there is a curve $C \subseteq X$ such that $C \cong \mathbb{P}^1$ and $C^2 = -1$, then there exists $f : X \to Y$ that exhibits X as a blow up of Y at a point with exceptional curve C.

Corollary. A smooth projective surface is relatively minimal if and only if it does not contain a (-1) curve. $\square$

3 Projective varieties

3.1 The intersection product

Lemma. Assume $D \equiv 0$. Then for all $D_2, \ldots, D_{\dim X}$, we have

$$D \cdot D_2 \cdot \ldots \cdot D_{\dim X} = 0.$$

Theorem (Severi)**.** Let X be a projective variety over an algebraically closed field. Then $N^1(X)$ is a finitely-generated torsion free abelian group, hence of the form $\mathbb{Z}^n$. □

Theorem (Asymptotic Riemann–Roch)**.** Let X be a projective normal variety over $K = \bar{K}$. Let D be a Cartier divisor, and E a Weil divisor on X. Then $\chi(X, \mathcal{O}_X(mD + E))$ is a numerical polynomial in m (i.e. a polynomial with rational coefficients that only takes integral values) of degree at most $n = \dim X$, and

$$\chi(X, \mathcal{O}_X(mD + E)) = \frac{D^n}{n!} m^n + \text{lower order terms.}$$

Proposition. Let X be a normal projective variety, and $|H|$ a very ample linear system. Then for a general element $G \in |H|$, G is a normal projective variety. □

Proposition. Let X be a normal projective variety.

(i) If H is a very ample Cartier divisor, then

$$h^0(X, mH) = \frac{H^n}{n!} m^n + \text{lower order terms for } m \gg 0.$$

(ii) If D is any Cartier divisor, then there is some $C \in \mathbb{R}_{>0}$ such that

$$h^0(mD) \leq C \cdot m^n \text{ for } m \gg 0.$$

3.2 Ample divisors

Lemma. Let X, Y be projective schemes. If $f : X \to Y$ is a finite morphism of schemes, and D is an ample Cartier divisor on Y, then so is f^*D.

Proposition. Let X be a proper scheme, and $\mathcal{L}$ an invertible sheaf. Then $\mathcal{L}$ is ample iff $\mathcal{L}|_{X_{\mathrm{red}}}$ is ample.

Theorem (Nakai's criterion)**.** Let X be a projective variety. Let D be a Cartier divisor on X. Then D is ample iff for all $V \subseteq X$ integral proper subvariety (proper means proper scheme, not proper subset), we have

$$(D|_V)^{\dim V} = D^{\dim V}[V] > 0.$$

Corollary. Let X be a projective variety. Then ampleness is a numerical condition, i.e. for any Cartier divisors D_1, D_2, if $D_1 \equiv D_2$, then D_1 is ample iff D_2 is ample.

Corollary. Let X, Y be projective variety. If $f : X \to Y$ is a surjective finite morphism of schemes, and D is a Cartier divisor on Y. Then D is ample iff f^*D is ample.

Corollary. If X is a projective variety, D a Cartier divisor and $\mathcal{O}_X(D)$ globally generated, and
$$\Phi : X \to \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*)$$
the induced map. Then D is ample iff Φ is finite.

Proposition. Let $D \in \mathrm{CaDiv}_{\mathbb{Q}}(X)$ Then the following are equivalent:

(i) cD is an ample integral divisor for some $c \in \mathbb{N}_{>0}$.

(ii) $D = \sum c_i D_i$, where $c_i \in \mathbb{Q}_{>0}$ and D_i are ample Cartier divisors.

(iii) D satisfies Nakai's criterion. That is, $D^{\dim V}[V] > 0$ for all integral subvarieties $V \subseteq X$.

Lemma. A positive linear combination of ample divisors is ample.

Proposition. Ampleness is an open condition. That is, if D is ample and $E_1, \ldots E_r$ are Cartier divisors, then for all $|\varepsilon_i| \ll 1$, the divisor $D + \varepsilon_i E_i$ is still ample.

Proposition. Being ample is a numerical property over $\mathbb{R}$, i.e. if $D_1, D_2 \in \mathrm{CaDiv}_{\mathbb{R}}(X)$ are such that $D_1 \equiv D_2$, then D_1 is ample iff D_2 is ample.

Proposition. Let H be an ample $\mathbb{R}$-divisor. Then for all $\mathbb{R}$-divisors $E_1, \ldots, E_r$, for all $\|\varepsilon_i\| \leq 1$, the divisor $H + \sum \varepsilon_i E_i$ is still ample. $\square$

3.3 Nef divisors

Proposition.

(i) D is nef iff $D|_{X_{\mathrm{red}}}$ is nef.

(ii) D is nef iff $D|_{X_i}$ is nef for all irreducible components X_i.

(iii) If $V \subseteq X$ is a proper subscheme, and D is nef, then $D|_V$ is nef.

(iv) If $f : X \to Y$ is a finite morphism of proper schemes, and D is nef on Y, then f^*D is nef on X. The converse is true if f is surjective. $\square$

Theorem (Kleinmann's criterion)**.** Let X be a proper scheme, and D an $\mathbb{R}$-Cartier divisor. Then D is nef iff $D^{\dim V}[V] \geq 0$ for all proper irreducible subvarieties.

Corollary. Let X be a projective scheme, and D be a nef $\mathbb{R}$-divisor on X, and H be a Cartier divisor on X.

(i) If H is ample, then $D + \varepsilon H$ is also ample for all $\varepsilon > 0$.

(ii) If $D + \varepsilon H$ is ample for all $0 < \varepsilon \ll 1$, then D is nef.

Corollary. $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$ and $\mathrm{int}(\mathrm{Nef}(X)) = \mathrm{Amp}(X)$.

Proposition. $\mathrm{Nef}(X) = \mathrm{NE}(X)^\vee$. $\square$

Proposition. $\overline{NE(X)} = \mathrm{Nef}(X)^\vee$. $\square$

Theorem (Kleinmann's criterion)**.** If X is a projective scheme and $D \subseteq \mathrm{CaDiv}_{\mathbb{R}}(X)$. Then the following are equivalent:

(i) D is ample

(ii) $D|_{\overline{NE(X)}} > 0$, i.e. $D \cdot \gamma > 0$ for all $\gamma \in \overline{NE(X)}$.

(iii) $\mathbb{S}_1 \cap \overline{NE(X)} \subseteq \mathbb{S}_1 \cap D_{>0}$, where $\mathbb{S}_1 \subseteq N_1(X)_{\mathbb{R}}$ is the unit sphere under some choice of norm.

Proposition. Let X be a projective scheme, and $D, H \in N^1_{\mathbb{R}}(X)$. Assume that H is ample. Then D is ample iff there exists $\varepsilon > 0$ such that

$$\frac{\mathrm{D} \cdot C}{H \cdot C} \geq \varepsilon.$$

Theorem (Cone theorem)**.** Let X be a smooth projective variety over $\mathbb{C}$. Then there exists rational curves $\{C_i\}_{i \in I}$ such that

$$\overline{\mathrm{NE}(X)} = \overline{\mathrm{NE}}_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+[C_i]$$

where $\overline{\mathrm{NE}}_{K_X \geq 0} = \{\gamma \in \overline{\mathrm{NE}(X)} : K_X \cdot \gamma \geq 0\}$. Further, we need at most countably many C_i's, and the accumulation points are all at $K_X^{\perp}$.

3.4 Kodaira dimension

Theorem (Iitaka)**.** Let X be a normal projective variety and $\mathcal{L}$ a line bundle on X. Suppose there is an m such that $|\mathcal{L}^{\otimes m}| \neq 0$. Then there exists X_∞, Y_∞, a map $\psi_\infty : X_\infty \to Y_\infty$ and a birational map $U_\infty : X_\infty \dashrightarrow X$ such that for $K \gg 0$ such that $|\mathcal{L}^{\otimes K}| \neq 0$, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, dashed, "\varphi_{|\mathcal{L}^{\otimes k}|}"] & \mathrm{Im}(\varphi_{|\mathcal{L}^{\otimes K}|}) \\
X_\infty \arrow[u, dashed, "U_\infty"] \arrow[r, "\psi_\infty"] & Y_\infty \arrow[u, dashed]
\end{tikzcd}$$

where the right-hand map is also birational.