EXAMPLE SHEET 1. POSITIVITY IN ALGEBRAIC GEOMETRY, L18

Instructions: This is the first example sheet. More exercises may be added during this week as the lectures and the topics I explain in class progress.

The first example class will be held on Tuesday Feb 27, 3-4.30pm. If you want, you can turn in the solutions of **up to 3** exercises from the example sheet. If you wish to do so, please turn your material in to my pigeonhole at CMS by Friday 23 9am. I will return your marked work in class. Exercises which are indicated with "Exercise*" are those for which I am suggesting writing down a proof.

Exercises that are denoted by "Exercise n.*" or "(n)*" are to be considered particularly challenging exercises.

Caveat: You should not expect to solve all these exercises at first sight. They are supposed to test your understanding of the material explained in class – sometimes in non-standard ways – and also help you develop a formally correct and complete way to put your ideas on paper.

By the word variety, we mean a reduced irreducible algebraic scheme (over a field). All varieties will be assumed to be over an algebraically closed field of characteristic 0 unless otherwise specified.

Exercise 1. *Let X be a quasi-projective variety over a field K. Let $U \subset X$ be an affine Zariski open set, $U \simeq \mathrm{Spec}(A)$, where A is a finitely generated K-algebra. Then $K(X) \simeq Q(A)$ the field of quotients of the of A.*

Exercise 2. *Let X, Y be two quasi-projective varieties over K. Then the following are equivalent:*

(1) X, Y are birational equivalent over K;
(2) there exists rational morphisms $\phi\colon X \dashrightarrow Y$, $\psi\colon Y \dashrightarrow X$ such that $\phi \circ \psi = \mathrm{Id}_Y$, $\psi \circ \phi = \mathrm{Id}_X$;
(3) there exists isomorphic Zariski open sets $U \subset X, V \subset Y$.

Exercise 3. *Let $C = V(X^2 + Y^2 + 1 = 0) \subset \mathbb{A}^2_{\mathbb{R}}$ be a conic without real point in the real plane. Show that C is not birationally equivalent to $\mathbb{P}^1_{\mathbb{R}}$.*
What happens if we consider the conic with the same equation inside $\mathbb{A}^2_{\mathbb{C}}$? Does it become birationally equivalent to $\mathbb{P}^2_{\mathbb{C}}$?
What happens if we change the equation of the conic to $X^2 + Y^2 - 1 = 0$?
How many smooth proper curves are there, up to isomorphism, over $\mathbb{R}$, which become isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ once we extend the field of definition?

Exercise 4. *Let X be a scheme over a noetherian ring A and let $\mathcal{L}$ be an invertible sheaf on X. If there exists $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ such that there exists a point $p \in X$ and $j \in \{0, \ldots, n\}$ s.t. $s_{j,p} \notin \mathfrak{m}_p \mathcal{L}_p$, then there exists a unique rational map defined over K, $\phi\colon X \dashrightarrow \mathbb{P}^n$. That is, there is a non-empty Zariski open set $U \subset X$ and a well defined morphism $\phi_U\colon U \to \mathbb{P}^n$.*

Exercise 5. *Let X be a projective scheme over $K = \overline{K}$.*

Let $\mathcal{L}$ be an invertible sheaf and let $V \subset H^0(X, \mathcal{L})$ be a subspace of sections separating closed points and tangent vectors on X.

(1) Let us fix two separate bases $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_n\}$. In what way are the two morphisms from X to $\mathbb{P}^n$ associated with the two different choices of bases related?

(2) *Let now $W \subsetneq V \subset H^0(X, \mathcal{L})$ be a subspace of V that separates closed points and tangent vectors on X. Choose a basis $\{r_0, \dots, r_i, r_{i+1}, \dots, r_n\}, 0 \leq i < n$ in such a way that $\{r_0, \dots, r_i\}$ is a basis of W. In what way are the two morphisms from $X \to \mathbb{P}^n,\ X \to \mathbb{P}^i$ associated with the two different choices of subspaces related?*

Exercise 6. (1) *Let $\mathcal{L}, \mathcal{M}$ be very ample invertible sheaves on a projective scheme X. Show that $\mathcal{L} \otimes \mathcal{M}$ is also very ample.*

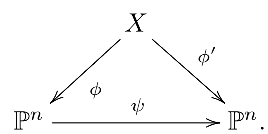
(2) *Show that the same holds true if we substitute very ample with ample.*

(3) *Let D be an ample Cartier divisor on a normal projective variety X. For any Cartier divisor E, there exists $m \in \mathbb{N}_{>0}$ s.t. $E + mD$ is ample.*

Exercise 7. *Let X be a normal projective variety over an algebraically closed field k. Show that there exists a $1-1$ correspondence in between*

$$\left\{ \begin{array}{c} \textit{non-degenerate morphisms} \\ \phi\colon X \to \mathbb{P}^n \\ \textit{up to projective equivalence} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (\mathcal{L}, |V|) \textit{ s.t. } |V| \textit{ is an } n-\textit{dimensional} \\ \textit{globally generated linear system of } \mathcal{L} \\ \textit{up to isomorphism} \end{array} \right\}.$$

Here up projective equivalence means up to an isomorphism of $\mathbb{P}^n$. That is, two maps $\phi\colon X \to \mathbb{P}^n,\ \phi'\colon X \to \mathbb{P}^n$ are to be identified if and only if there exists an isomorphism $\psi\colon \mathbb{P}^n \to \mathbb{P}^n$ s.t. the following diagram commutes



Non-degenerate means that the image of X is not contained in a proper linear subspace of $\mathbb{P}^n$.

Analogously, two pairs $(\mathcal{L}, |V|), (\mathcal{L}', |V'|)$ are to be identified if and only if there exists an isomorphism of invertible sheaf $\chi\colon \mathcal{L} \to \mathcal{L}'$ s.t. at the level of sections χ maps $|V|$ isomorphically onto $|V'|$.

Exercise* 8. *Show that, analogously, we have a map between*

$$\left\{ \begin{array}{c} (\mathcal{L}, |V|) \textit{ s.t. } |V| \textit{ is an } n-\textit{dimensional} \\ \textit{linear system of the invertible sheaf } \mathcal{L} \\ \textit{up to isomorphism} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{non-denerate rational morphisms} \\ \phi\colon X \dashrightarrow \mathbb{P}^n \\ \textit{up to projective equivalence} \end{array} \right\}$$

Is this map injective? Surjective? Try to give a proof/construct examples in both case.

Exercise 9. *A twisted cubic $X \subset \mathbb{P}^3$ is a rational curve (i.e. isomorphic to $\mathbb{P}^1$) of degree 3 in three-dimensional projective space, which is non-degenerate, i.e. it is not contained in a linear subspace of $\mathbb{P}^3$.*

Show that all twisted cubics are projectively equivalent.

Exercise 10. *Let C be a proper smooth curve over a field k. Show that a divisor D on C is ample if and only if $\deg D > 0$.*

Exercise 11. *Let Q be the smooth quadric surface qhose equation is given by $V(x_0x_1 - x_2x_3) \subset \mathbb{P}^3$.*

(1) *Show that $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and that its Picard group is isomorphic to Z^2 and it is generated by $p_1^*\mathcal{O}_{\mathbb{P}^1}(1), p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ where p_i is the projection onto the i-th factor. In particular, we say that an invertible sheaf $\mathcal{L}$ has degree (a,b) if and only if $\mathcal{L} \simeq p_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(b)$.*
(2) *Show that invertible sheaves of degree $(a,b) \in \mathbb{Z}^2$ are ample iff $a, b > 0$.*
(3) *Compute the intersection form on the Picard group of Q.*
(4) *Compute an orthogonal basis for the form of intersection on $\mathrm{Pic}(Q)$.*
(5) *For which $(a,b) \in \mathbb{Z}^2$, an invertible sheaf of degree (a,b) has sections? Let $\mathcal{L}$ one such invertible sheaf: how does $h^0(Q, \mathcal{L}^m)$ grow as a function of m?*

Exercise 12. *Let X be a smooth projective surface. Assume that $\mathcal{L}$ be an invertible sheaf such that $H^0(X, \mathcal{L}^m) \geq Cm^2$, for some $C > 0$ and $\forall m \gg 0$.*

(1) *Then there exists m_0 s.t. $|\mathcal{L}_0^m|$ induces a rational map $X \dashrightarrow \mathbb{P}^n$ which is birational onto its image.*
(2)* *If for every proper curve $C \subset X$, $\deg \mathcal{L}|_C > 0$ then $\mathcal{L}$ is ample.*

Exercise 13. (1) *Determine the genus of a smooth degree d curve $X \subset \mathbb{P}^2$.*
(2) *Can you do the same when $X \subset \mathbb{P}^{n+1}$ is a complete intersection of hypersurfaces $H_1, \ldots, H_n$ of degree $d_1, \ldots, d_n$ respectively.*

Exercise 14. *Let X be a projective variety of dimension n. Let $\mathcal{L}$ be a very ample invertible sheaf on X.*

(1) *Show that for $m \gg 0, h^0(X, \mathcal{L}^m) = P(m)$ for a degree n polynomial P with coefficients in $\mathbb{Q}$.*
(2) *Can you describe the leading coefficient of P in terms of geometric data of X?*
(3) *Does the same conclusion hold if $\mathcal{L}$ is just ample?*

Perhaps you may want to start working out hte case where $\dim X = 1$. *(Hint: you may want to use* [Har, Prop. I.7.3]*.)*

Exercise 15. *Let C, D be any two divisors on a surface X and let $\mathcal{L}, \mathcal{M}$ be the corresponding invertible sheaves. Show that*

$$C \cdot D = \chi(\mathcal{O}_X) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{M}^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{M}^{-1}).$$

Exercise* 16. *Let D be a divisor on a smooth projective surface X s.t. $D^2 > 0$, then mD is effective for some $0 \neq m \in \mathbb{Z}$*

Exercise 17. (1) *Let C be an irreducible curve over a smooth projective surface X s.t. $C^2 < 0$. Show that $h^0(X, \mathcal{O}_X(mC)) = 1$, $\forall m \in \mathbb{N}$. Does the conclusion still hold if we remove the condition that C is irreducible?*
(2) *Let D be a divisor on a smooth surface such that $h^0(X, \mathcal{O}_X(mD)) = O(m^2)$, i.e., $\exists C \in \mathbb{R}_{>0}$ s.t. $h^0(X, \mathcal{O}_X(mD)) \geq Cm^2$, $\forall m \gg 0$. Then there exists an ample divisor H and an effective divisor E s.t. for some $n \in \mathbb{N}_{>0}$, $nD \sim H + E$.*

For the purpose of the next couple exercises, let us recall the fact that a divisor D on a smooth projective variety X is nef if $D \cdot C \geq 0$ for any curve $C \subset X$.

Exercise 18. *Let X be a surface for which $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = 0$. Prove that if D is a nef divisor with $D^2 = 0 \not\equiv D$ then D is effective.*

Is D a semiample divisor?

Exercise 19. *Let $X_n = \mathbb{F}_n$ be the n-th Hirzebruch surface.*

(1) *Show that* $\rho(X_n) = 2, \forall n$.
(2) *Can you describe the set of all nef divisors on* X_n*?*
(3) *What are the effective divisors with negative self-intersection?*

Exercise 20. *Let* $X = V(xt - yz) \subset \mathbb{A}^4$ *be the quadric hypersurface which is the affine cone over a quadric in* $\mathbb{P}^3$.

(1) *Show that* X *is normal.*
(2)* *Show that the plane* $l = V(x, z) \subset X$ *is not a Cartier divisor in* X*, and none of its multiple can be.*

Exercise 21. *Assume that* $C, D \subset X$ *are curves without common components in a smooth projective surface.*

Then show that $C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_p$*, where* $(C \cdot D)_p$ *is defined in the following way: let* f, g *be local equations of* C, D *respectively at* $p \in X$*; then* $(C \cdot D)_p = \dim_k \mathcal{O}_{X,p}/(f, g)$*, where* $\dim_k$ *indicates the dimension as a* k*-vector space.*

Exercise 22. *Generalize the canonical bundle formula to the case of a codimension* 1 *smooth subvariety* $Y \subset X$ *of a smooth variety* X*. What if the codimension of* Y *is greater than* 1 *(yet, both* X *and* Y *are still smooth)?*

References

[Har] R. Hartshorne, *Algebraic Geometry*. Springer, GTM 52, 1997. 3

Example sheet 2. Positivity in Algebraic Geometry, L18

Instructions: This is the second example sheet. More exercises may be added during this week as the lectures and the topics I explain in class progress.

The second example class will be held on Tuesday Mar 13, 3-4.30pm in MR6. If you want, you can turn in the solutions of **up to 5** exercises from the example sheet. If you wish to do so, please turn your material in to my pigeonhole at CMS by Monday Mar 12, 9am. I will return your marked work in class. Exercises which are indicated with "Exercise*" are those for which I am suggesting writing down a proof.

Exercises that are denoted by "Exercise n.*" or "(n)*" are to be considered particularly challenging exercises.

Caveat: You should not expect to solve all these exercises at first sight. They are supposed to test your understanding of the material explained in class – sometimes in non-standard ways – and also help you develop a formally correct and complete way to put your ideas on paper.

By the word variety, we mean a reduced irreducible algebraic scheme (over a field). All varieties will be assumed to be over an algebraically closed field of characteristic 0 unless otherwise specified.

Exercise 1. *Let X be a projective scheme. Then verify that the definition of the intersection pairing given as follows*

$$D_1 \cdot \dots \cdot D_n = \prod_{i \in I \subset \{1,\dots,n\}} ((-1)^{n-|I|} \prod_{i \in I} H_{i,1} \prod_{j \in \{1,\dots,n\} \setminus I} H_{j,2}),$$
$$D_i \sim H_{i,1} - H_{i,2},\ H_{i,j} \text{ very ample,}$$

is independent of the choice of the very ample divisors H_i^j used to rewrite any Cartier divisor in the definition as difference of Cartier divisors. (Hint: proceed as in the analogous check that we did for the 2-dimensional case)

Exercise 2. *Let X be a projective variety. Then show that the definition of the intersection product in terms of Euler characteristic,*

$$D_1 \cdot \dots \cdot D_n = \sum_{I \subset \{1,\dots,n\}} (-1)^{|I|} \chi(X, \mathcal{O}_X(\sum_{i \in I} -D_i)),$$

is well posed and respects all the needed properties (1)-(4') stated in lectures.

(Hint: to prove linearity, you may want to prove the following fact:

Let $E, F, D_1, \dots, D_{\dim X}$ be Cartier divisors. Then

$$\sum_{I \subset \{2,\dots,n\}} (-1)^{|I|} (\chi(X, \sum_{i \in I} -D_i) - \chi(X, -E + \sum_{i \in I} -D_i) -$$
$$-\chi(X, -F + \sum_{i \in I} -D_i) + \chi(X, -E - F + \sum_{i \in I} -D_i) = 0.$$

I suggest to try and prove this statement by induction on the dimension of X, using the usual reduction about very ample divisors.)

Exercise 3. *Let X be a projective scheme. Let H be a very ample divisor on X. Then $H^n = \deg_H(X)$, where $\deg_H(X)$ indicates the degree of X with respect to the embedding of X given by $|H|$.*

Exercise 4. *Let X be a proper algebraic variety.*

1. *Assume that any Weil divisor $D \in \mathrm{Div}(X)$ is Cartier, then show that the intersection product is a symmetric multilinear form defined over $\mathrm{N}^1(X)$.*
2. *Assume that for any Weil divisor $D \in \mathrm{Div}(X)$ there exists $m \in \mathbb{N}_{>0}$ s.t. mD is Cartier. Can you still define the intersection product on $\mathrm{N}^1(X)$? If yes, how?*

Exercise* 5.

1. *Is it possible to prove the Asymptotic Riemann-Roch formula in the case where X is just an irreducible projective variety? What if X is just a projective scheme?*
2. *Is it possible to identify the second term in the asymptotic Riemann-Roch formula above? That is, if we write*
$$\chi(X, \mathcal{O}_X(mD+E)) = \frac{D^n}{n!}m^n + b_{n-1}m^{n-1} + O(m^{n-2}),$$
what is b_{n-1}? Try to give an answer in terms of intersection numbers.
(Hint: use Riemann-Roch for surfaces to get a first inductive step and then try to argue as in the proof of the asymptotic Riemann-Roch formula).
3. *Prove the following generalised form of Asymptotic Riemann-Roch.*
Let X be an irreducible projective variety. Let D be a Cartier divisor and $\mathcal{F}$ be a coherent sheaf on X.
Then the Euler characteristic of $\mathcal{F}(mD) = \mathcal{F} \otimes \mathcal{O}_X(mD)$ is a polynomial of degree $\leq \dim X$ in m with rational coefficients. More precisely, it has the following (asymptotic) form:
$$\chi(X, \mathcal{F}(mD)) = \mathrm{rank}(\mathcal{F}) \cdot \frac{D^n}{n!}m^n + O(m^{n-1}).$$

Exercise 6. *Let X_d be a degree d hypersurface in $\mathbb{P}^n$.*
Compute $P_t(m) = \chi(X_d, \mathcal{O}_{X_d}(mt))$.

Exercise 7. *Let $\mathbb{F}_1$ be the blow-up of $\mathbb{P}^2$ at one point.*
Recall that $\mathbb{F}_1$ has a natural fibration $\pi \colon \mathbb{F}_1 \to \mathbb{P}^1$. Let F indicate a fibre of π, $F \simeq \mathbb{P}^1$. Let E denote the exceptional curve $E \simeq \mathbb{P}^1, E^2 = -1$ for the blow-up.

1. *Show that all effective divisors on $\mathbb{F}_1$ are linearly equivalent to $aE + bF$ for some choice of $a, b \in \mathbb{N}^2$.*
2. *Compute the asymptotic Riemann-Roch formula for divisors of the above form. What can you say about the growth-rate of the higher cohomology groups of such divisors?*

Exercise 8. *Generalize the result in the previous proposition to the case of a proper scheme X.*
(Hint: use Chow's Lemma).

Exercise 9. *Let X be a proper variety and let D be a Cartier divisor s.t. $\mathcal{O}_X(D)$ is generated by global sections.*
Let $\phi \colon X \to \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*)$ be the map induced by the linear system of $|D|$.
Then $D^{\dim X} > 0$ if and only if ϕ is a generically finite morphism – that is, the generic fibre is 0-dimensional.

Exercise 10. *Show that if $g \colon V' \to V$ is a proper finite surjective map of irreducible varieties and D is a Cartier divisor on V then*
$$(g^*D)^{\dim V'} = \deg g \cdot D^{\dim V}.$$

Exercise 11. *Let $f\colon X \to Y$ be a finite surjective morphism and let $V \subset Y$ be a proper irreducible subvariety of Y. Then there exists a proper irreducible subvariety $V' \subset f^{-1}(V) \subset X$ s.t. the restriction of f to V', $f|_{V'}\colon V \to V$ is a finite surjective morphism onto V.*

Exercise 12. *Show an example where $f\colon X \to Y$ is a finite map of projective schemes, D is a Cartier divisor on Y s.t. f^*D is ample on X, but D is not ample on X.*

Exercise 13. *Let X be a proper scheme. Let D be a Cartier $\mathbb{R}$-divisor.*

(1) D is nef if and only if $D|_{X_{\mathrm{red}}}$ is nef.
(2) If X is reducible nad $X = \cup_i X_i$ are its irreducible components, then D is nef if and only if $D|_{X_i}$ is nef $\forall i$.
(3) If $V \subset X$ is a proper subscheme. If D is nef, then $D|_V$ is nef.

Exercise* 14. *(1) Let $C \subset X$ be an irreducible curve on a smooth projective surface X with $C^2 > 0$. Then $\mathcal{O}_X(mC)$ is globally generated for $m \gg 0$. How does the morphism induced by $|mC|$ look like for $m \gg 0$?*
(2) Is it true that an irreducible curve $C \subset X$ with ample normal bundle is ample? Give either a proof of this statement or a counterxample. If you can find a counterexample, explain what condition could be added to this statement that would imply ampleness.
(3) Let X be a smooth projective variety and let $D \subset X$ an effective divisor such with ample normal bundle. Show that $\mathcal{O}_X(mD)$ is globally generated for $m \gg 0$. How does the morphism induced by $|mC|$ look like for $m \gg 0$?

Exercise 15. *(1) Prove that the intersection pairing*

$$\mathrm{N}^1(X)_K \times \mathrm{N}_1(X)_K \longrightarrow K \qquad (1)$$

$$(D,\ E) \longmapsto D \cdot E.$$

is perfect over $\mathbb{Q}, \mathbb{R}$.
(2) Show that if we consider the pairing in (1) only for $\mathrm{N}^1(X)$ and $\mathrm{N}_1(X)$ the pairing may not be perfect, i.e. we don't necessarily get that $\mathrm{N}^1(X) \simeq \mathrm{N}_1(X)^*$.
Give an example where this happens.
Show that, nonetheless, the intersection pairing gives an embedding of $\mathrm{N}^1(X)$ as a finite index subgroup of $\mathrm{N}_1(X)^*$.

Exercise 16. *Let X be a smooth surface and let D be a nef divisor such that $D^2 = 0 \not\equiv D$.*

(1) Let C be a curve on X such that $C^2 \geq 0, D \cdot C = 0$. Then D and C are parallel in $\mathrm{N}_1(X)_{\mathbb{R}}$, that is, there exists $\lambda \in \mathbb{R}_{>0}$ s.t. $D - \lambda C \equiv 0$.
(2) Assume that there exists a variety T, a projective morphism $p\colon S \to T$ and one $q\colon S \to X$ s.t. the fibers of p are all reduced and irreducible curves and their image under q are curves along which D has degree 0. Show that $\dim T \leq 1$.

Exercise 17. *(1) Show that $\mathrm{Nef}(X)$ (resp. $\mathrm{NE}(X)$) do not contain any linear subspaces of $\mathrm{N}^1(X)_{\mathbb{R}}$ (resp. $\mathrm{N}_1(X)_{\mathbb{R}}$).*

(2) *Let* $C \subset V$, $C' \subset V^*$ *be closed cones in finite dimensional* $\mathbb{R}$- *vector spaces. Assume that* $C^\vee = C'$. *Then* C' *is closed and* $C'^\vee = \overline{C}$.
(3) *Same notation as in the previous part. If* $D \in \operatorname{relint}(C)$ *then* $D|_{C' \setminus 0} > 0$.
(4) $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X)^\vee$.
(5) *Let* D *be an ample* $\mathbb{R}$-*divisor. Show that* $D|_{\overline{\mathrm{NE}}(X) \setminus 0} > 0$.