Part III — Positivity in Algebraic Geometry
Definitions

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This class aims at giving an introduction to the theory of divisors, linear systems and their positivity properties on projective algebraic varieties.

The first part of the class will be dedicated to introducing the basic notions and results regarding these objects and special attention will be devoted to discussing examples in the case of curves and surfaces.

In the second part, the course will cover classical results from the theory of divisors and linear systems and their applications to the study of the geometry of algebraic varieties.

If time allows and based on the interests of the participants, there are a number of more advanced topics that could possibly be covered: Reider's Theorem for surfaces, geometry of linear systems on higher dimensional varieties, multiplier ideal sheaves and invariance of plurigenera, higher dimensional birational geometry.

Pre-requisites

The minimum requirement for those students wishing to enroll in this class is their knowledge of basic concepts from the Algebraic Geometry Part 3 course, i.e. roughly Chapters 2 and 3 of Hartshorne's Algebraic Geometry.

Familiarity with the basic concepts of the geometry of algebraic varieties of dimension 1 and 2 — e.g. as covered in the preliminary sections of Chapters 4 and 5 of Hartshorne's Algebraic Geometry — would be useful but will not be assumed — besides what was already covered in the Michaelmas lectures.

Students should have also some familiarity with concepts covered in the Algebraic Topology Part 3 course such as cohomology, duality and characteristic classes.

Contents

1 Divisors

1.1 Projective embeddings

Definition (Generating section). Let X be a scheme, and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Let $s_0, \ldots, s_n \in H^0(X, \mathcal{F})$ be sections. We say the sections *generate* $\mathcal{F}$ if the natural map

$$\bigoplus_{i=0}^{n+1} \mathcal{O}_X \to \mathcal{F}$$

induced by the s_i is a surjective map of $\mathcal{O}_X$-modules.

Definition (Very ample sheaf). Let X be an algebraic variety over K, and $\mathcal{L}$ be an invertible sheaf. We say that $\mathcal{L}$ is very ample if there is a closed immersion $\varphi : X \to \mathbb{P}^n$ such that $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{L}$.

Definition (Separate points and tangent vectors). With the hypothesis of the proposition, we say that

- elements of V *separate points* if V satisfies (i).
- elements of V *separate tangent vectors* if V satisfies (ii).

Definition (Ample sheaf). Let X be a Noetherian scheme over A, and $\mathcal{L}$ an invertible sheaf over X. We say $\mathcal{L}$ is ample iff for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, there is an n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections.

1.2 Weil divisors

Definition (Regular in codimension 1). Let X be a Noetherian scheme. We say X is *regular in codimension 1* if every local ring $\mathcal{O}_x$ of dimension 1 is regular.

Definition (Weil divisor). Let X be a Noetherian scheme, regular in codimension 1. A *prime divisor* is a codimension 1 integral subscheme of X. A *Weil divisor* is a formal sum

$$D = \sum a_i Y_i,$$

where the $a_i \in \mathbb{Z}$ and Y_i are prime divisors. We write $\mathrm{WDiv}(X)$ for the group of Weil divisors of X.

For K a field, a *Weil K-divisor* is the same where we allow $a_i \in K$.

Definition (Effective divisor). We say a Weil divisor is *effective* if $a_i \geq 0$ for all i. We write $D \geq 0$.

Definition (Principal divisor). If $f \in K(X)$, then we define the *principal divisor*

$$\mathrm{div}(f) = \sum_Y \mathrm{val}_Y(f) \cdot Y.$$

Definition (Support). The *support* of $D = \sum a_i Y_i$ is

$$\mathrm{supp}(D) = \bigcup Y_i.$$

Definition (Class group). The *class group* of X, $\mathrm{Cl}(X)$, is the group of Weil divisors quotiented out by the principal divisors.

We say Weil divisors D, D' are *linearly equivalent* if $D - D'$ is principal, and we write $D \sim D'$.

1.3 Cartier divisors

Definition (Locally principal)**.** Let D be a Weil divisor on X. Fix $x \in X$. Then D is locally principal at x if there exists an open set $U \subseteq X$ containing x such that $D|_U = \operatorname{div}(f)|_U$ for some $f \in K(X)$.

Definition (Cartier divisor)**.** A *Cartier divisor* is a locally principal Weil divisor.

Definition (Picard group)**.** We define the *Picard group* of X to be the group of Cartier divisors modulo principal equivalence.

1.4 Computations of class groups

1.5 Linear systems

Definition (Complete linear system)**.** A *complete linear system* is the set of all effective divisors linearly equivalent to a given divisor D_0, written $|D_0|$.

Definition (Linear system)**.** A linear system is a linear subspace of the projective space structure on $|D_0|$.

Definition ((Very) ample divisor)**.** We say a Cartier divisor D is *(very) ample* when $\mathcal{O}_X(D)$ is.

2 Surfaces

2.1 The intersection product

Definition (Intersection product)**.** For divisors D_1, D_2, we define the *intersection product* to be

$$D_1 \cdot D_2 = \chi(\mathcal{O}_X) + \chi(-D_1 - D_2) - \chi(-D_1) - \chi(-D_2).$$

Definition (Numerical equivalence)**.** We say divisors D, D' are *numerically equivalent*, written $D \equiv D'$, if

$$D \cdot E = D' \cdot E$$

for all divisors E.

We write

$$\mathrm{Num}_0 = \{D \in \mathrm{Div}(X) : D \equiv 0\}.$$

Definition (Néron–Severi group)**.** The *Néron–Severi group* is

$$\mathrm{NS}(X) = \mathrm{Div}(X)/\mathrm{Num}_0(X).$$

Definition ($\rho(X)$)**.** $\rho(X) = \dim \mathrm{NS}_{\mathbb{R}}(X) = \mathrm{rk}\,\mathrm{NS}(X)$.

2.2 Blow ups

Definition (Blow up)**.** The *blow up* of X at p is the variety $\bar{X} = \mathrm{Bl}_p X$ that we just defined. The preimage of p is known as the *exceptional curve*, and is denoted E.

Definition (Relatively minimal)**.** We say X is *relatively minimal* if there does not exist a smooth X' and a birational morphism $X \to X'$ that is not an isomorphism. We say X is *minimal* if it is the unique relative minimal surface in the birational equivalence class.

3 Projective varieties

3.1 The intersection product

Definition (Numerical equivalence). Let $D, D' \in \mathrm{CaDiv}(X)$. We say D and D' are *numerically equivalent*, written $D \equiv D'$, if $D \cdot [C] = D' \cdot [C]$ for all integral curves $C \subseteq X$.

Definition (Neron–Severi group). The *Neron–Severi group* of X, written $\mathrm{NS}(X) = N^1(X)$, is

$$N^1(X) = \frac{\mathrm{CaDiv}(X)}{\mathrm{Num}_0(X)} = \frac{\mathrm{CaDiv}(X)}{\{D \mid D \equiv 0\}}.$$

Definition (Picard number). The *Picard number* of X is the rank of $N^1(X)$.

3.2 Ample divisors

3.3 Nef divisors

Definition (Numerically effective divisor). Let X be a proper scheme, and D a Cartier divisor. Then D is *numerically effective (nef)* if $D \cdot C \geq 0$ for all integral curves $C \subseteq X$.

Definition (1-cycles). For $K = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, we define the space of 1-cycles to be

$$Z_1(X)_K = \left\{\sum a_i C_i : a_i \in K, C_i \subseteq X \text{ integral proper curves}\right\}.$$

Definition (Numerical equivalence). Let X be a proper scheme, and $C_1, C_2 \in Z_1(X)_K$. Then $C_1 \equiv C_2$ iff

$$D \cdot C_1 = D \cdot C_2$$

for all $D \in N^1_K(X)$.

Definition ($N_1(X)_K$). We define $N_1(X)_K$ to be the K-module of K 1-cycles modulo numerical equivalence.

Definition (Effective curves). We define the cone of effective curves to be

$$\mathrm{NE}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} : \gamma \equiv \sum [a_i C_i] : a_i > 0, C_i \subseteq X \text{ integral curves}\}.$$

Definition (Dual cone). Let V be a finite-dimensional vector space over $\mathbb{R}$ and V^* the dual of V. If $C \subseteq V$ is a cone, then we define the *dual cone* $C^\vee \subseteq V^*$ by

$$C^\vee = \{f \in V^* : f(c) \geq 0 \text{ for all } c \in C\}.$$

3.4 Kodaira dimension

Definition (Kodaira dimension). The *Kodaira dimension* of $\mathcal{L}$ is

$$K(X, \mathcal{L}) = \begin{cases} -\infty & h^0(X, \mathcal{L}^{\otimes m}) = 0 \text{ for all } m > 0 \\ \dim(Y_\infty) & \text{otherwise} \end{cases}.$$