Modular forms and L-functions (Lent 2017) — example sheet #1

Comments/corrections to a.j.scholl@dpmms.cam.ac.uk

1. Prove that if $f(x) = e^{-\pi x^2}$ then $\widehat{f}(x) = f(x)$.

2. Let G be a finite abelian group, and χ, χ' characters of G. Show that

$$\sum_{g \in G} \overline{\chi(g)} \chi'(g) = \begin{cases} 0 & \text{if } \chi \neq \chi' \\ \#G & \text{if } \chi = \chi' \end{cases}$$

3. Show that every continuous homomorphism $\chi\colon \mathbb{R}^\times_{>0} \to \mathbb{C}^\times$ is of the form $\chi(x) = x^s$ for some $s \in \mathbb{C}$. [Hint: first describe all continuous homomorphisms $\mathbb{R} \to \mathbb{C}$.] Show also that $\widehat{\mathbb{R}} = \{\chi_y \colon x \mapsto e^{2\pi i x y} \mid y \in \mathbb{R}\}$.

4. Show that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \Lambda(n) n^{-s}$$

where Λ is the *Von Mangoldt function*:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ k \geq 1,\ p \text{ prime} \\ 0 & \text{if } n = 1 \text{ or } n \text{ is not a prime power} \end{cases}$$

Show also that $\sum_{d|n} \Lambda(d) = \log n$.

5. Evaluate $\zeta(2k)$ is terms of Bernoulli numbers. Deduce that $(-1)^{k-1} B_{2k} > 0$ for every $k \geq 1$. (It is not easy to prove this directly from the generating function definition!)

6. A subgroup $\Lambda \subset \mathbb{R}^n$ is *discrete* if is is discrete as a topological space (i.e. for any ball $B \subset \mathbb{R}^n$, $\Lambda \cap B$ is finite). Show that Λ is discrete iff $\Lambda = \sum \mathbb{Z} x_i$ where $\{x_i\} \subset \mathbb{R}^n$ is an $\mathbb{R}$-linearly independent set.

Modular forms and L-functions (Lent 2017) — example sheet #2

Comments/corrections to `a.j.scholl@dpmms.cam.ac.uk`

1. (i) The *Bernoulli polynomials* $B_n(X)$ are defined by the formula

$$\sum_{k=0}^{\infty} B_k(X)\frac{t^k}{k!} = \frac{te^{tX}}{e^t - 1}.$$

Let $\psi\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be any periodic function. Use the analytic continuation to write the values at negative integers of the L-series $L(\psi, s)$ in terms of the values $B_k(j/N)$.

(ii) Consider the (inverse) Fourier transform

$$\widehat{B}_{n,N}(\zeta) = \sum_{j=0}^{N-1} \zeta^j B_n(j/N) \qquad (\zeta^N = 1)$$

Show that if $\zeta \neq 1$, then $\widetilde{B}_n(\zeta) \stackrel{\text{def}}{=} N^{1-n}B_{n,N}(\zeta) = P_n(\zeta)/(\zeta - 1)^n$ for some polynomial P_n which does not depend on N. (The substitution $u = e^{-t/N}$ may be useful.)

(iii) Obtain for $D > 1$ the *distribution relation*: if $\zeta^N = 1$ then

$$\sum_{\eta^D=\zeta} \widetilde{B}_n(\eta) = D^n \widetilde{B}_n(\zeta)\,.$$

2. Fix a positive real number D. Let $\mathcal{X}_D$ be the set of real symmetric 2×2 matrices of determinant D. Let $SL_2(\mathbb{R})$ act on $\mathcal{X}_D$ by $g\colon X \to gXg^t$ (g^t = transpose of g). Describe the orbits of this action, and identify one of them with the upper halfplane.

3. Write $E_6(z)\Delta(z) = \sum_{n=1}^{\infty} c_n q^n$. Show that $c_n \equiv \sigma_{17}(n) \pmod{43867}$. Obtain similar congruences for the coefficients of $E_4\Delta$, $E_8\Delta$, $E_{10}\Delta$ and $E_{14}\Delta$.

[NB: $B_{16} = -3617/510$, $B_{18} = 43867/798$, $B_{20} = -174611/330$, $B_{22} = 854513/138$, $B_{26} = 8553103/6$.]

4. (i) Let f be a modular function (= a weakly modular form of weight 0). Show that $\mathrm{ord}_{\tau=i}\, f \equiv 0 \pmod 2$ and $\mathrm{ord}_{\tau=\rho}\, f \equiv 0 \pmod 3$.

(ii) Let $f \in M_k$. Show that if $k \not\equiv 0 \pmod 4$ then $f(i) = 0$, and that if $k \not\equiv 0 \pmod 3$ then $f(\rho) = 0$, where $\rho = e^{\pi i/3}$.

5. Let $f \in M_k$ and $g \in M_l$ be modular forms. Show that $lf'g - kfg' \in M_{k+l+2}$.

6. Let $f\colon \mathcal{H} \to \mathbb{C}$ satisfy $f|_k\gamma = f$ for all $\gamma \in \Gamma(1)$. Show that $y^{k/2}\,|f(x+iy)|$ is invariant under $z = x + iy \mapsto \gamma(z)$. Show that if moreover f is holomorphic on $\mathcal{H}$ and $k > 0$, then f is a cusp form if and only if $y^{k/2}\,|f|$ is bounded on $\mathcal{H}$ (or equivalently, is bounded on $\mathcal{D}$).

7. Define

$$G_2(z) = \sum_{m=-\infty}^{\infty}{}' \left(\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^2} \right)$$

where the inner sum is over all integers n, except where $m = 0$, in which case the term $n = 0$ is omitted. By rewriting the inner sum, show that the series converges to

$$\frac{\pi^2}{3}\left(1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n\right).$$

Explain why $G_2(z)$ is not a modular form of weight 2. (Later we will prove that G_2 satisfies a somewhat more complicated transformation law for $z \mapsto -1/z$.)

Modular forms and L-functions (Lent 2017) — example sheet #3

Comments/corrections to a.j.scholl@dpmms.cam.ac.uk

(In these questions, Γ denotes a finite index subgroup of $\Gamma(1)$.)

1. Fix an even weight $k \geq 4$. Let $\mathbb{T} \subset \operatorname{End} S_k$ be the subalgebra of endomorphisms generated over $\mathbb{Z}$ by the Hecke operators T_n, $n \geq 1$. Let $S_k(\mathbb{Z}) = S_k(\Gamma(1), \mathbb{Z})$ denote the submodule of cusp forms of level 1 and weight k with integral Fourier coefficients. Show that $S_k(\mathbb{Z})$ is stable under $\mathbb{T}$, and that the map

$$\begin{aligned} S_k(\mathbb{Z}) \times \mathbb{T} &\to \mathbb{Z} \\ (f, T_n) &\mapsto a_1(T_n f) \end{aligned}$$

gives an isomorphism between $S_k(\mathbb{Z})$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{Z})$, which is an isomorphism of $\mathbb{T}$-modules. (Hint consider the basis $\{g_j \mid 1 \leq j \leq m = \dim S_k\}$ for which $a_n(g_j) = \delta_{jn}$ for $1 \leq n \leq m$.)

2. Let (G, Γ) satisfy property (H). Suppose there is a map $\sigma \colon G \to G$ such that

   (i) $\sigma^2(g) = g$ and $\sigma(gh) = \sigma(h)\sigma(g)$ for all g, $h \in G$; and

   (ii) σ fixes every double coset $\Gamma g \Gamma$.

   Show that the Hecke algebra of (G, Γ) is then commutative. Use this to give another proof of the commutativity of the Hecke algebra for $(GL_2(\mathbb{Q})^+, SL_2(\mathbb{Z}))$.

3. Let $\Lambda \subset \mathbb{R}^k$ be a lattice (i.e. a discrete subgroup of maximal rank). Say that Λ is *self-dual* if the set

$$\Lambda^* = \{x \in \mathbb{R}^k \mid x \cdot y \text{ is an integer for every } y \in \Lambda\}$$

is precisely Λ, and that Λ is *even* if $\|x\|^2 \in 2\mathbb{Z}$ for every $x \in \Lambda$. (Here $\|-\|$ denotes Euclidean norm.)

Use the Poisson summation formula to show that if $\Lambda \subset \mathbb{R}^k$ is an even self-dual lattice, then the theta series

$$\theta_\Lambda(z) = \sum_{x \in \Lambda} \exp(\pi i \|x\|^2 z)$$

is a modular form of weight $k/2$ and level 1. (In particular, k is divisible by 4.)

4. Let k be a positive integer divisible by 4. Let Λ be the set of all $x = (x_1 \ldots, x_k) \in \mathbb{R}^k$ satisfying

$$2x_i \in \mathbb{Z}, \qquad x_i - x_j \in \mathbb{Z}, \qquad \sum_{i=1}^{k} x_i \in 2\mathbb{Z}.$$

Show that Λ is a self-dual lattice. [Λ is usually denoted E_k.]

(ii) Suppose further that k is divisible by 8. Show that Λ is even.

(iii) Finally let $k = 8$. Show that $\theta_\Lambda(z) = E_4(z)$. Hence (or directly) show that there are exactly 240 elements $x \in \Lambda$ with $\|x\|^2 = 2$.

5. Let $p > 2$ be prime. Draw a fundamental domain for $\Gamma^0(p)$ as given in the lectures, and show that the identifications of points along the boundary are given as follows:

   - the vertical lines $\operatorname{Im}(z) = \pm p/2$ are identified by the translation $z \mapsto z + p$.
   - the circular arcs $C_a = \{|z - a| = 1\}$, for integers a with $0 < |a| < p/2$, are identified as follows: C_a is identified with C_b iff $ab \equiv -1 \pmod{p}$.

6. (i) Show that Γ has a fundamental domain which is a *connected* union of translates of the standard fundamental domain for $\Gamma(1)$.

   (ii) * Show that if Γ has no elements of finite order other than ± 1 then Γ is a free group. (Use the fact that any group acting without fixed points on a tree is free.)

7. (i) Show that $\Gamma(N)$ is torsionfree if $N \geq 3$ and that the only elements of finite order of $\Gamma(2)$ are $\{\pm 1\}$.

   (ii) Show that if $N \geq 4$ then $\Gamma_1(N)$ is torsionfree.

8. Let
$$\Gamma^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Gamma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

   (i) Show that if $f \in M_k(\Gamma)$, then the function $f^*(z) = \overline{f(-\bar{z})}$ belongs to $M_k(\Gamma^*)$.

   (ii) Show that if $\Gamma = \Gamma^*$ (for example, any one of $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$) then $M_k(\Gamma)$ has a basis all of whose elements have real Fourier coefficients.

9. (i) Let ν be the number of cusps of Γ. Show that there is a linear map $M_k(\Gamma) \to \mathbb{C}^\nu$ whose kernel is $S_k(\Gamma)$.

   (ii) By considering the Eisenstein series $E_{\underline{r},k}$ for suitable $\underline{r} \in N^{-1}\mathbb{Z}^2$, show that if $k > 2$ then there exists $f \in M_k(\Gamma(N))$ whose q-expansion at ∞ has contant term 1, but which vanishes at all other cusps of $\Gamma(N)$. Deduce that $\dim M_k(\Gamma(N)) = \nu + \dim S_k(\Gamma(N))$ if $k \geq 3$.

10. (i) Show that if every cusp of Γ has width one then Γ must be $\Gamma(1)$.

    (ii) * Show that if Γ is a congruence subgroup containing -1, then $\Gamma \supset \Gamma(N)$ where N is the least common multiple of the widths of the cusps of Γ. (This gives a way to tell whether or not a given group is a congruence subgroup.)