Part III — Logic

Theorems

Based on lectures by T. E. Forster

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

This course is the sequel to the Part II courses in Set Theory and Logic and in Automata and Formal Languages lectured in 2015-6. (It is already being referred to informally as "Son of ST&L and Automata & Formal Languages"). Because of the advent of that second course this Part III course no longer covers elementary computability in the way that its predecessor ("Computability and Logic") did, and this is reflected in the change in title. It will say less about Set Theory than one would expect from a course entitled 'Logic'; this is because in Lent term Benedikt Löwe will be lecturing a course entitled 'Topics in Set Theory' and I do not wish to tread on his toes. Material likely to be covered include: advanced topics in first-order logic (Natural Deduction, Sequent Calculus, Cut-elimination, Interpolation, Skolemisation, Completeness and Undecidability of First-Order Logic, Curry-Howard, Possible world semantics, Gödel's Negative Interpretation, Generalised quantifiers. . .); Advanced Computability (λ-representability of computable functions, Tennenbaum's theorem, Friedberg-Muchnik, Baker-Gill-Solovay. . .); Model theory background (ultraproducts, Los's theorem, elementary embeddings, omitting types, categoricity, saturation, Ehrenfeucht-Mostowski theorem. . .); Logical combinatorics (Paris-Harrington, WQO and BQO theory at least as far as Kruskal's theorem on wellquasiorderings of trees. . .). This is a new syllabus and may change in the coming months. It is entirely in order for students to contact the lecturer for updates.

Pre-requisites

The obvious prerequisites from last year's Part II are Professor Johnstone's Set Theory and Logic and Dr Chiodo's Automata and Formal Languages, and I would like to assume that everybody coming to my lectures is on top of all the material lectured in those courses. This aspiration is less unreasonable than it may sound, since in 2016-7 both these courses are being lectured the term before this one, in Michaelmas; indeed supervisions for Part III students attending them can be arranged if needed: contact me or your director of studies. I am lecturing Part II Set Theory and Logic and I am even going to be issuing a "Sheet 5" for Set Theory and Logic, of material likely to be of interest to people who are thinking of pursuing this material at Part III. Attending these two Part II courses in Michaelmas is a course of action that may appeal particularly to students from outside Cambridge.

Contents

1 Proof theory and constructive logic

1.1 Natural deduction

1.2 Curry–Howard correspondence

Proposition. We cannot prove $((A \to B) \to A) \to A$ in natural deduction without the law of excluded middle.

1.3 Possible world semantics

Lemma. Any formula with a natural deduction proof not using the rule for classical negation is true in all possible world models.

1.4 Negative interpretation

Lemma. Any formula built up from negated and doubly negated atomics by $\neg$, $\wedge$ and $\forall$ is stable.

1.5 Constructive mathematics

2 Model theory

2.1 Universal theories

Lemma. Let T be a consistent theory, and let $T_\forall$ be the set of all universal consequences of T, i.e. all things provable from T that are of the form $(*)$. Let $\mathcal{M}$ be a model of $T_\forall$. Then $T \cup D(\mathcal{M})$ is also consistent.

Theorem. A theory T is universal if and only if every substructure of a model of T is a model of T.

2.2 Products

Proposition. Products preserve (universal) Horn formulae.

Theorem (Łoś theorem)**.** Let $\{A_i : i \in I\}$ be a family of structures of the same (first-order) signature, and $\mathcal{U} \subseteq P(I)$ an ultrafilter. Then

$$\prod_{i \in I} A_i/\mathcal{U} \vDash \varphi \Longleftrightarrow \{i : A_i \vDash \varphi\} \in \mathcal{U}.$$

In particular, if A_i are all models of some theory, then so is $\prod A_i/\mathcal{U}$.

Lemma. Let F be a filter on I. Then the following are equivalent:

(i) F is an ultrafilter.

(ii) For $X \subseteq I$, either $X \in F$ or $I \setminus X \in F$ ("F is prime").

(iii) If $X, Y \subseteq I$ and $X \cup Y \in I$, then $X \in I$ or $Y \in I$.

Theorem (Compactness theorem)**.** Let T be a theory in first order logic such that every finite subset has a model. Then T has a model.

2.3 Ehrenfeucht–Mostowski theorem

Proposition. Monadic first-order logic is decidable.

Theorem (Ehrenfeucht–Mostowski theorem (1956))**.** Let $\langle I, \leq \rangle$ be a total order, and let T be a theory with infinite models. Suppose we have a unary predicate P and a 2-ary relation $\preccurlyeq \in \mathcal{L}(T)$ such that

$$T \vdash \preccurlyeq \text{ is a total order on } \{x : P(x)\}.$$

Then T has a model $\mathcal{M}$ with a copy of I as a sub-order of $\preccurlyeq$, and the copy of I is a set of indiscernibles. Moreover, we can pick $\mathcal{M}$ such that every order-automorphism of $\langle I, \leq \rangle$ extends to an automorphism of $\mathcal{M}$.

2.4 The omitting type theorem

Theorem (Omitting type theorem)**.** Let T be a first-order theory, and Σ an n-type. If

$$T \vdash \forall_x \neg \varphi(x)$$

whenever φ locally realizes Σ, then T has a model omitting Σ.

Theorem. Let T be a propositional theory, and $\Sigma \subseteq \mathcal{L}(T)$ a type (with $n = 0$). If T locally omits Σ, then there is a T-valuation that omits Σ.

Theorem. Let T be a propositional theory, and for each $i \in \mathbb{N}$, we let $\Sigma_i \subseteq \mathcal{L}(T)$ be types for each $i \in \mathbb{N}$. If T locally omits each Σ_i, then there is a T-valuation omitting all Σ_i.

3 Computability theory

3.1 Computability

Proposition. There exists primitive recursion functions $\mathsf{pair} : \mathbb{N}^2 \to \mathbb{N}$ and $\mathsf{unpair} : \mathbb{N} \to \mathbb{N}^2$ such that

$$\mathsf{unpair}(\mathsf{pair}(x, y)) = (x, y)$$

for all $x, y \in \mathbb{N}$.

Corollary. There exists $\mathsf{cons} : \mathbb{N} \to \mathbb{N}, \mathsf{head} : \mathbb{N} \to \mathbb{N}$ and $\mathsf{tail} : \mathbb{N} \to \mathbb{N}$ such that

$$\mathsf{cons}(\mathsf{head}\ x, \mathsf{tail}\ x) = x$$

Proposition. The Ackermann function is well-defined.

Theorem. The function $n \mapsto A(n, n)$ dominates all primitive recursive functions.

3.2 Decidable and semi-decidable sets

Proposition. A set $X \subseteq \mathbb{N}^k$ is semi-decidable iff it is a projection of a decidable subset of $\mathbb{N}^{k+1}$.

Theorem (Turing)**.** The halting set is semi-decidable but not decidable.

Theorem (*smn* theorem)**.** There is a total computable function s of two variables such that for all e, we have

$$\{e\}(b, a) = \{s(e, b)\}(a).$$

Similarly, we can find such an s for any tuples $\mathbf{b}$ and $\mathbf{a}$.

Theorem (Fixed point theorem)**.** Let $h : \mathbb{N} \to \mathbb{N}$ be total computable. Then there is an $n \in \mathbb{N}$ such that $\{n\} = \{h(n)\}$ (as functions).

Theorem (Rice's theorem)**.** Let A be a non-empty proper subset of the set of all computable functions $\mathbb{N} \to \mathbb{N}$. Then $\{n : \{n\} \in A\}$ is not decidable.

Corollary. It is impossible to grade programming homework.

3.3 Computability elsewhere

3.4 Logic

Theorem (Craig's theorem)**.** Every first-order theory with a semi-decidable set of axioms has a decidable set of axioms.

Theorem (Tennenbaum's theorem)**.** For any countable non-standard model of true arithmetic, the graph of $+$ and $\times$ cannot be decidable.

Theorem. The set of Gödel numbers of machines that compute total functions is not semi-decidable.

Theorem (Gödel's incompleteness theorem)**.** Let T be a recursively axiomatized theory of arithmetic, i.e. the set of axioms is semi-decidable (hence decidable). Suppose T is sufficiently strong to talk about computability of functions (e.g. Peano arithmetic is). Then there is some proposition true in $\mathbb{N}$ that cannot be proven in T.

Theorem (Ramsey's theorem)**.** We write $\mathbb{N}^{(k)}$ for the set of all subsets of $\mathbb{N}$ of size k. Suppose we partition $\mathbb{N}^{(k)}$ in m many distinct pieces. Then there exists some infinite $X \subseteq \mathbb{N}$ such that X is monochromatic, i.e. $X^{(k)} \subseteq \mathbb{N}^{(k)}$ lie entirely within a partition.

Theorem (Jockusch)**.** There exists a decidable partition of $\mathbb{N}^{(3)}$ into two pieces with no infinite decidable monochromatic set.

3.5 Computability by λ-calculus

Theorem. Every λ expression can be reduced to at most one β-normal form. Moreover, there exists a reduction strategy such that whenever a λ expression can be reduced to a β-normal form, then it will be reduced to it via this reduction strategy.

This magic reduction strategy is just to always perform β-reduction on the leftmost thing possible.

Theorem. For any g, we have

$$\mathsf{Y}\ g \rightsquigarrow g\ (\mathsf{Y}\ g).$$

3.6 Reducibility

Proposition. Let $A \subseteq \mathbb{N}$. Then $A \leq_m \mathbb{K}$ iff A is semi-decidable.

Proposition. $(\mathbb{N} \setminus \mathbb{K}) \leq_M A$ iff A is productive.

Theorem (Friedberg–Muchnik)**.** There exists two $A, B \subseteq \mathbb{N}$ such that $A \not\leq B \not\leq A$. Moreover, A and B are both semi-decidable.

4 Well-quasi-orderings

Axiom (Axiom of dependent choice)**.** Let X be a set an R a relation on X. The *axiom of dependent choice* says if for all $x \in X$, there exists $y \in X$ such that $R(x, y)$, then we can find a sequence $x_1, x_2, \cdots$ such that $R(x_i, x_{i+1})$ for all $i \in \mathbb{N}$.

Lemma. If $\langle X, \leq_X \rangle$ is a well-founded quasi-order, then so is $X^{<\omega}$ and $\text{Trees}(X)$.

Proposition. Let $\langle X, \leq$ be a quasi-order. Then the following are equivalent:

(i) $\langle X, \leq \rangle$ is a well-quasi-order.

(ii) There is no infinite decreasing sequence and no infinite anti-chain.

(iii) Whenever we have any sequence $x_i \in X$ whatsoever, we can find $i < j$ such that $x_i \leq x_j$.

Lemma. Let $\langle X, \leq \rangle$ be a well-founded quasi-order that is not a WQO. Then it has a minimal bad sequence.

Lemma (Minimal bad sequence lemma)**.** Let $\langle X, \leq \rangle$ be a quasi-order and $B = \{b_i\}$ a minimal bad sequence. Let

$$X' = \{x \in X : \exists_{n \in \mathbb{N}} x < b_n\}.$$

Then $\langle X', \leq \rangle$ is a WQO.

Lemma (Perfect subsequence lemma)**.** Let $\langle X, \leq \rangle$ be a WQO. Then every sequence $\{x_n\}$ in X has a *perfect subsequence*, i.e. an infinite subset $A \subseteq \mathbb{N}$ such that for all $i < j \in A$, we have $f(i) \leq f(j)$.

Lemma (Higman's lemma)**.** Let $\langle X, \leq_X \rangle$ be a WQO. Then $\langle X^{<\omega}, \leq_s \rangle$ is a WQO.

Theorem (Kruskal's theorem)**.** Let $\langle X, \leq_X \rangle$ be a WQO. Then $\langle \text{Trees}(X), \leq_s \rangle$ is a WQO.

Proposition (Friedman's finite form)**.** For all $k \in \mathbb{N}$, the statement $P(k)$ is true.