# Part III — Algebras
## Theorems with proof

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The aim of the course is to give an introduction to algebras. The emphasis will be on non-commutative examples that arise in representation theory (of groups and Lie algebras) and the theory of algebraic D-modules, though you will learn something about commutative algebras in passing.

Topics we discuss include:

- Artinian algebras. Examples, group algebras of finite groups, crossed products. Structure theory. Artin–Wedderburn theorem. Projective modules. Blocks. $K_0$.
- Noetherian algebras. Examples, quantum plane and quantum torus, differential operator algebras, enveloping algebras of finite dimensional Lie algebras. Structure theory. Injective hulls, uniform dimension and Goldie's theorem.
- Hochschild chain and cochain complexes. Hochschild homology and cohomology. Gerstenhaber algebras.
- Deformation of algebras.
- Coalgebras, bialgebras and Hopf algebras.

### Pre-requisites

It will be assumed that you have attended a first course on ring theory, eg IB Groups, Rings and Modules. Experience of other algebraic courses such as II Representation Theory, Galois Theory or Number Fields, or III Lie algebras will be helpful but not necessary.

# Contents

# 0 Introduction

**Theorem** (Artin–Wedderburn theorem)**.** Let $A$ be a left-Artinian algebra such that the intersection of the maximal left ideals is zero. Then $A$ is the direct sum of finitely many matrix algebras over division algebras.

**Theorem** (Goldie's theorem)**.** Let $A$ be a right Noetherian algebra with no non-zero ideals all of whose elements are nilpotent. Then $A$ embeds in a finite direct sum of matrix algebras over division algebras.

# 1 Artinian algebras

## 1.1 Artinian algebras

**Proposition.** Let $A$ be an algebra and $I$ a left ideal. Then $I$ is a maximal left ideal iff $A/I$ is simple.

**Proposition.** Let $A$ be an algebra and $M$ a simple module. Then $M \cong A/I$ for some (maximal) left ideal $I$ of $A$.

*Proof.* Pick an arbitrary element $m \in M$, and define the $A$-module homomorphism $\varphi : A \to M$ by $\varphi(a) = am$. Then the image is a non-trivial submodule, and hence must be $M$. Then by the first isomorphism theorem, we have $M \cong A/\ker\varphi$. □

**Lemma.** Let $M$ be a finitely-generated $A$ module. Then $M$ has a maximal proper submodule $M'$.

*Proof.* Let $m_1, \cdots, m_k \in M$ be a minimal generating set. Then in particular $N = \langle m_1, \cdots, m_{k-1}\rangle$ is a proper submodule of $M$. Moreover, a submodule of $M$ containing $N$ is proper iff it does not contain $m_k$, and this property is preserved under increasing unions. So by Zorn's lemma, there is a maximal proper submodule. □

**Lemma** (Nakayama lemma)**.** The following are equivalent for a left ideal $I$ of $A$.

(i) $I \leq J(A)$.

(ii) For any finitely-generated left $A$-module $M$, if $IM = M$, then $M = 0$, where $IM$ is the module generated by elements of the form $am$, with $a \in I$ and $m \in M$.

(iii) $G = \{1 + a : a \in I\} = 1 + I$ is a subgroup of the unit group of $A$.

*Proof.*

- (i) ⇒ (ii): Suppose $I \leq J(A)$ and $M \neq 0$ is a finitely-generated $A$-module, and we'll see that $IM \lneq M$.

  Let $N$ be a maximal submodule of $M$. Then $M/N$ is a simple module, so for any $\bar{m} \in M/N$, we know $\mathrm{Ann}(\bar{m})$ is a maximal left ideal. So $J(A) \leq \mathrm{Ann}(M/N)$. So $IM \leq J(A)M \leq N \lneq M$.

- (ii) ⇒ (iii): Assume (ii). We let $x \in I$ and set $y = 1 + x$. Hence $1 = y - x \in Ay + I$. Since $Ay + I$ is a left ideal, we know $Ay + I = A$. In other words, we know
  $$I\left(\frac{A}{Ay}\right) = \frac{A}{Ay}.$$
  Now using (ii) on the finitely-generated module $A/Ay$ (it is in fact generated by 1), we know that $A/Ay = 0$. So $A = Ay$. So there exists $z \in A$ such that $1 = zy = z(1 + x)$. So $(1 + x)$ has a left inverse, and this left inverse $z$ lies in $G$, since we can write $z = 1 - zx$. So $G$ is a subgroup of the unit group of $A$.

– (iii) $\Rightarrow$ (i): Suppose $I_1$ is a maximal left ideal of $A$. Let $x \in I$. If $x \notin I_1$, then $I_1 + Ax = A$ by maximality of $I$. So $1 = y + zx$ for some $y \in I_1$ and $z \in A$. So $y = 1 - zx \in G$. So $y$ is invertible. But $y \in I_1$. So $I_1 = A$. This is a contradiction. So we found that $I < I_1$, and this is true for all maximal left ideals $I_1$. Hence $I \leq J(A)$. □

**Proposition.** Let $M$ be an $A$-module. Then the following are equivalent:

(i) $M$ is completely reducible.

(ii) $M$ is the direct sum of simple modules.

(iii) Every submodule of $M$ has a *complement*, i.e. for any submodule $N$ of $M$, there is a complement $N'$ such that $M = N \oplus N'$.

*Proof.*

– (i) $\Rightarrow$ (ii): Let $M$ be completely reducible, and consider the set

$$\left\{\{S_\alpha \leq M\} : S_\alpha \text{ are simple, } \sum S_\alpha \text{ is a direct sum}\right\}.$$

Notice this set is closed under increasing unions, since the property of being a direct sum is only checked on finitely many elements. So by Zorn's lemma, it has a maximal element, and let $N$ be the sum of the elements.

Suppose this were not all of $M$. Then there is some $S \leq M$ such that $S \not\leq N$. Then $S \cap N \subsetneq S$. By simplicity, they intersect trivially. So $S + N$ is a direct sum, which is a contradiction. So we must have $N = M$, and $M$ is the direct sum of simple modules.

– (ii) $\Rightarrow$ (i) is trivial.

– (i) $\Rightarrow$ (iii): Let $N \leq M$ be a submodule, and consider

$$\left\{\{S_\alpha \leq M\} : S_\alpha \text{ are simple, } N + \sum S_\alpha \text{ is a direct sum}\right\}.$$

Again this set has a maximal element, and let $P$ be the direct sum of those $S_\alpha$. Again if $P \oplus N$ is not all of $M$, then pick an $S \leq M$ simple such that $S$ is not contained in $P \oplus N$. Then again $S \oplus P \oplus N$ is a direct sum, which is a contradiction.

– (iii) $\Rightarrow$ (i): It suffices to show that if $N < M$ is a proper submodule, then there exists a simple module that intersects $N$ trivially. Indeed, we can take $N$ to be the sum of all simple submodules of $M$, and this forces $N = M$.

To do so, pick an $x \notin N$, and let $P$ be submodule of $M$ maximal among those satisfying $P \cap N = 0$ and $x \notin N \oplus P$. Then $N \oplus P$ is a proper submodule of $M$. Let $S$ be a complement. We claim $S$ is simple.

If not, we can find a proper submodule $S'$ of $S$. Let $Q$ be a complement of $N \oplus P \oplus S'$. Then we can write

$$\begin{array}{ccccccccc} M & = & N & \oplus & P & \oplus & S' & \oplus & Q \\ x & = & n & + & p & + & s & + & q \end{array}.$$

By assumption, $s$ and $q$ are not both zero. We wlog assume $s$ is non-zero. Then $P \oplus Q$ is a larger submodule satisfying $(P \oplus Q) \cap N = 0$ and $x \notin N \oplus (P \oplus Q)$. This is a contradiction. So $S$ is simple, and we are done. □

**Proposition.** Sums, submodules and quotients of completely reducible modules are completely reducible.

*Proof.* It is clear by definition that sums of completely reducible modules are completely reducible.

To see that submodules of completely reducible modules are completely reducible, let $M$ be completely reducible, and $N \leq M$. Then for each $x \in N$, there is some simple submodule $S \leq M$ containing $x$. Since $S \cap N \leq S$ and contains $x$, it must be $S$, i.e. $S \subseteq N$. So $N$ is the sum of simple modules.

Finally, to see quotients are completely reducible, if $M$ is completely reducible and $N$ is a submodule, then we can write

$$M = N \oplus P$$

for some $P$. Then $M/N \cong P$, and $P$ is completely reducible. □

**Proposition.** Let $M$ be an $A$-module satisfying the descending chain condition on submodules. Then $M$ is completely reducible iff $\mathrm{Rad}(M) = 0$.

*Proof.* It is clear that if $M$ is completely reducible, then $\mathrm{Rad}(M) = 0$. Indeed, we can write

$$M = \bigoplus_{\alpha \in A} S_\alpha,$$

where each $S_\alpha$ is simple. Then

$$J(A) \leq \bigcap_{\alpha \in A} \left( \bigoplus_{\beta \in A \setminus \{\alpha\}} S_\beta \right) = \{0\}.$$

Conversely, if $\mathrm{Rad}(M) = 0$, we note that since $M$ satisfies the descending chain condition on submodules, there must be a *finite* collection $M_1, \cdots, M_n$ of maximal submodules whose intersection vanish. Then consider the map

$$\begin{aligned} M &\longrightarrow \bigoplus_{i=1}^n \frac{M}{M_i} \\ x &\longmapsto (x + M_1, x + M_2, \cdots, x + M_n) \end{aligned}$$

The kernel of this map is the intersection of the $M_i$, which is trivial. So this embeds $M$ as a submodule of $\bigoplus \frac{M}{M_i}$. But each $\frac{M}{M_i}$ is simple, so $M$ is a submodule of a completely reducible module, hence completely reducible. □

**Corollary.** If $A$ is a semi-simple left Artinian algebra, then ${}_A A$ is completely reducible.

**Corollary.** If $A$ is a semi-simple left Artinian algebra, then every left $A$-module is completely reducible.

*Proof.* Every $A$-module $M$ is a quotient of sums of ${}_AA$. Explicitly, we have a map

$$\begin{aligned}\bigoplus_{m \in M} {}_AA &\longrightarrow M\\ (a_m) &\longmapsto \sum a_m m\end{aligned}$$

Then this map is clearly surjective, and thus $M$ is a quotient of $\bigoplus_M {}_AA$. □

**Lemma.** Let $A$ be left Artinian, and $M$ a finitely generated left $A$-module, then $J(A)M = \mathrm{Rad}(M)$.

*Proof.* Let $M'$ be a maximal submodule of $M$. Then $M/M'$ is simple, and is in fact $A/I$ for some maximal left ideal $I$. Then we have

$$J(A)\left(\frac{M}{M'}\right) = 0,$$

since $J(A) < I$. Therefore $J(A)M \leq M'$. So $J(A)M \leq \mathrm{Rad}(M)$.

Conversely, we know $\frac{M}{J(A)M}$ is an $A/J(A)$-module, and is hence completely reducible as $A/J(A)$ is semi-simple (and left Artinian). Since an $A$-submodule of $\frac{M}{J(A)M}$ is the same as an $A/J(A)$-submodule, it follows that it is completely reducible as an $A$-module as well. So

$$\mathrm{Rad}\left(\frac{M}{J(A)M}\right) = 0,$$

and hence $\mathrm{Rad}(M) \leq J(A)M$. □

**Proposition.** Let $A$ be left Artinian. Then

(i) $J(A)$ is nilpotent, i.e. there exists some $r$ such that $J(A)^r = 0$.

(ii) If $M$ is a finitely-generated left $A$-module, then it is both left Artinian and left Noetherian.

(iii) $A$ is left Noetherian.

*Proof.*

(i) Since $A$ is left-Artinian, and $\{J(A)^r : r \in \mathbb{N}\}$ is a descending chain of ideals, it must eventually be constant. So $J(A)^r = J(A)^{r+1}$ for some $r$. If this is non-zero, then again using the descending chain condition, we see there is a left ideal $I$ with $J(A)^r I \neq 0$ that is minimal with this property (one such ideal exists, say $J(A)$ itself).

Now pick $x \in I$ with $J(A)^r x \neq 0$. Since $J(A)^{2r} = J(A)^r$, it follows that $J(A)^r(J(A)^r x) \neq 0$. So by minimality, $J(A)^r x \geq I$. But the other inclusion clearly holds. So they are equal. So there exists some $a \in J(A)^r$ with $x = ax$. So

$$(1-a)x = 0.$$

But $1 - a$ is a unit. So $x = 0$. This is a contradiction. So $J(A)^r = 0$.

(ii) Let $M_i = J(A)^i M$. Then $M_i/M_{i+1}$ is annihilated by $J(A)$, and hence completely reducible (it is a module over semi-simple $A/J(A)$). Since $M$ is a finitely generated left $A$-module for a left Artinian algebra, it satisfies the descending chain condition for submodules (exercise), and hence so does $M_i/M_{i+1}$.

So we know $M_i/M_{i+1}$ is a finite sum of simple modules, and therefore satisfies the ascending chain condition. So $M_i/M_{i+1}$ is left Noetherian, and hence $M$ is (exercise).

(iii) Follows from (ii) since $A$ is a finitely-generated left $A$-module. □

## 1.2 Artin–Wedderburn theorem

**Theorem** (Artin–Wedderburn theorem)**.** Let $A$ be a semisimple right Artinian algebra. Then

$$A = \bigoplus_{i=1}^{r} M_{n_i}(D_i),$$

for some division algebra $D_i$, and these factors are uniquely determined.

$A$ has exactly $r$ isomorphism classes of simple (right) modules $S_i$, and

$$\mathrm{End}_A(S_i) = \{A\text{-module homomorphisms } S_i \to S_i\} \cong D_i,$$

and

$$\dim_{D_i}(S_i) = n_i.$$

If $A$ is simple, then $r = 1$.

**Lemma** (Schur's lemma)**.** Let $M_1, M_2$ be simple right $A$-modules. Then either $M_1 \cong M_2$, or $\mathrm{Hom}_A(M_1, M_2) = 0$. If $M$ is a simple $A$-module, then $\mathrm{End}_A(M)$ is a division algebra.

*Proof.* A non-zero $A$-module homomorphism $M_1 \to M_2$ must be injective, as the kernel is submodule. Similarly, the image has to be the whole thing since the image is a submodule. So this must be an isomorphism, and in particular has an inverse. So the last part follows as well. □

**Lemma.**

(i) If $M$ is a right $A$-module and $e$ is an idempotent in $A$, i.e. $e^2 = e$, then $Me \cong \mathrm{Hom}_A(eA, M)$.

(ii) We have

$$eAe \cong \mathrm{End}_A(eA).$$

In particular, we can take $e = 1$, and recover $\mathrm{End}_A(A_A) \cong A$.

*Proof.*

(i) We define maps

$$\begin{aligned} me &\longmapsto (ex \mapsto mex) \\ Me &\underset{f_2}{\overset{f_1}{\rightleftarrows}} \mathrm{Hom}(eA, M) \\ \alpha(e) &\longleftarrow\!\shortmid\ \alpha \end{aligned}$$

We note that $\alpha(e) = \alpha(e^2) = \alpha(e)e \in Me$. So this is well-defined. By inspection, these maps are inverse to each other. So we are done.

Note that we might worry that we have to pick representatives $me$ and $ex$ for the map $f_1$, but in fact we can also write it as $f(a)(y) = ay$, since $e$ is idempotent. So we are safe.

(ii) Immediate from above by putting $M = eA$. $\square$

**Lemma.** Let $M$ be a completely reducible right $A$-module. We write

$$M = \bigoplus S_i^{n_i},$$

where $\{S_i\}$ are distinct simple $A$-modules. Write $D_i = \mathrm{End}_A(S_i)$, which we already know is a division algebra. Then

$$\mathrm{End}_A(S_i^{n_i}) \cong M_{n_i}(D_i),$$

and

$$\mathrm{End}_A(M) = \bigoplus M_{n_i}(D_i)$$

*Proof.* The result for $\mathrm{End}_A(S_i^{n_i})$ is just the familiar fact that a homomorphism $S^n \to S^m$ is given by an $m \times n$ matrix of maps $S \to S$ (in the case of vector spaces over a field $k$, we have $\mathrm{End}(k) \cong k$, so they are matrices with entries in $k$). Then by Schur's lemma, we have

$$\mathrm{End}_A(M) = \bigoplus_i \mathrm{End}_A(M_i) \cong M_{n_i}(D_i). \qquad \square$$

*Proof of Artin–Wedderburn.* If $A$ is semi-simple, then it is completely reducible as a right $A$-module. So we have

$$A \cong \mathrm{End}(A_A) \cong \bigoplus M_{n_i}(D_i).$$

We now decompose each $M_{n_i}(D_i)$ into a sum of simple modules. We know each $M_{n_i}(D_i)$ is a non-trivial $M_{n_i}(D_i)$ module in the usual way, and the action of the other summands is trivial. We can simply decompose each $M_{n_i}(D_i)$ as the sum of submodules of the form

$$\left\{\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_2 & \cdots & a_{n_i-1} & a_{n_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\right\}$$

and there are $n_i$ components. We immediately see that if we write $S_i$ for this submodule, then we have

$$\dim_{D_i}(S_i) = n_i.$$

Finally, we have to show that every simple module $S$ of $A$ is one of the $S_i$. We simply have to note that if $S$ is a simple $A$-module, then there is a non-trivial map $f : A \to S$ (say by picking $x \in S$ and defining $f(a) = xa$). Then in the decomposition of $A$ into a direct sum of simple submodules, there must be one factor $S_i$ such that $f|_{S_i}$ is non-trivial. Then by Schur's lemma, this is in fact an isomorphism $S_i \cong S$. $\square$

**Corollary.** If $k$ is algebraically closed and $A$ is a finite-dimensional semi-simple $k$-algebra, then
$$A \cong \bigoplus M_{n_i}(k).$$

**Theorem** (Maschke's theorem)**.** Let $G$ be a finite group and $p \nmid |G|$, where $p = \operatorname{char} k$, so that $|G|$ is invertible in $k$, then $kG$ is semi-simple.

*Proof.* We show that any submodule $V$ of a $kG$-module $U$ has a complement. Let $\pi : U \to V$ be any $k$-vector space projection, and define a new map
$$\pi' = \frac{1}{|G|} \sum_{g \in G} g \pi g^{-1} : U \to V.$$
It is easy to see that this is a $kG$-module homomorphism $U \to V$, and is a projection. So we have
$$U = V \oplus \ker \pi',$$
and this gives a $kG$-module complement. □

**Theorem.** Let $G$ be finite and $kG$ semi-simple. Then $\operatorname{char} k \nmid |G|$.

*Proof.* We note that there is a simple $kG$-module $S$, given by the trivial module. This is a one-dimensional $k$ vector space. We have
$$D = \operatorname{End}_{kG}(S) = k.$$
Now suppose $kG$ is semi-simple. Then by Artin–Wedderburn, there must be only one summand of $S$ in $kG$.

Consider the following two ideals of $kG$: we let
$$I_1 = \left\{\sum \lambda_g g \in kG : \sum \lambda_g = 0\right\}.$$
This is in fact a two-sided ideal of $kG$. We also have the center of the algebra, given by
$$I_2 = \left\{\lambda \sum g \in kG : \lambda \in k\right\}.$$
Now if $\operatorname{char} k \mid |G|$, then $I_2 \subseteq I_1$. So we can write
$$kG = \frac{kG}{I_1} \oplus I_1 = \frac{kG}{I_1} \oplus I_2 \oplus \cdots.$$
But we know $G$ acts trivially on $\frac{kG}{I_1}$ and $I_2$, and they both have dimension 1. This gives a contradiction. So we must have $\operatorname{char} k \nmid |G|$. □

**Theorem.** Let $k$ be algebraically closed of characteristic $p$, and $G$ be finite. Then the number of simple $kG$ modules (up to isomorphism) is equal to the number of conjugacy classes of elements of order not divisible by $p$. These are known as the *$p$-regular elements*.

**Corollary.** If $|G| = p^r$ for some $r$ and $p$ is prime, then the trivial module is the only simple $kG$ module, when $\operatorname{char} k = p$.

*Proof sketch of theorem.* The number of simple $kG$ modules is just the number of simple $kG/J(kG)$ module, as $J(kG)$ acts trivially on every simple module. There is a useful trick to figure out the number of simple $A$-modules for a given semi-simple $A$. Suppose we have a decomposition

$$A \cong \bigoplus_{i=1}^{r} M_{n_i}(k).$$

Then we know $r$ is the number of simple $A$-modules. We now consider $[A, A]$, the $k$-subspace generated by elements of the form $xy - yx$. Then we see that

$$\frac{A}{[A, A]} \cong \bigoplus_{i=1}^{r} \frac{M_{n_i}(k)}{[M_{n_i}(k), M_{n_i}(k)]}.$$

Now by linear algebra, we know $[M_{n_i}(k), M_{n_i}(k)]$ is the trace zero matrices, and so we know

$$\dim_k \frac{M_{n_i}(k)}{[M_{n_i}(k), M_{n_i}(k)]} = 1.$$

Hence we know

$$\dim \frac{A}{[A, A]} = r.$$

Thus we need to compute

$$\dim_k \frac{kG/J(kG)}{[kG/J(kG), kG/J(kG)]}$$

We then note the following facts:

(i) For a general algebra $A$, we have

$$\frac{A/J(A)}{[A/J(A), A/J(A)]} \cong \frac{A}{[A, A] + J(A)}.$$

(ii) Let $g_1, \cdots, g_m$ be conjugacy class representatives of $G$. Then

$$\{g_i + [kG, kG]\}$$

forms a $k$-vector space basis of $kG/[kG, kG]$.

(iii) If $g_1, \cdots, g_r$ is a set of representatives of $p$-regular conjugacy classes, then

$$\Big\{g_i + \Big([kG, kG] + J(kG)\Big)\Big\}$$

form a basis of $kG/([kG, kG] + J(kG))$.

Hence the result follows. $\square$

## 1.3 Crossed products

## 1.4 Projectives and blocks

**Lemma.** The following are equivalent:

(i) $P$ is projective.

(ii) Every surjective map $\phi : M \to P$ splits, i.e.

$$M \cong \ker \phi \oplus N$$

where $N \cong P$.

(iii) $P$ is a direct summand of a free module.

*Proof.*

– (i) $\Rightarrow$ (ii): Consider the following lifting problem:

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, equal] & \\ M \arrow[r, two heads, "\phi"] & P \arrow[r] & 0 \end{tikzcd},$$

The lifting gives an embedding of $P$ into $M$ that complements $\ker \phi$ (by the splitting lemma, or by checking it directly).

– (ii) $\Rightarrow$ (iii): Every module admits a surjection from a free module (e.g. the free module generated by the elements of $P$)

– (iii) $\Rightarrow$ (i): It suffices to show that direct summands of projectives are projective. Suppose $P$ is is projective, and

$$P \cong A \oplus B.$$

Then any diagram

$$\begin{tikzcd} & A \arrow[d, "\alpha"] & \\ M' \arrow[r, two heads, "\theta"] & M \arrow[r] & 0 \end{tikzcd},$$

can be extended to a diagram

$$\begin{tikzcd} & A \oplus B \arrow[d, "\tilde{\alpha}"] & \\ M' \arrow[r, two heads, "\theta"] & M \arrow[r] & 0 \end{tikzcd},$$

by sending $B$ to 0. Then since $A \oplus B \cong P$ is projective, we obtain a lifting $A \oplus B \to M'$, and restricting to $A$ gives the desired lifting. □

**Theorem** (Krull–Schmidt theorem)**.** Suppose $M$ is a finite sum of indecomposable $A$-modules $M_i$, with each $\mathrm{End}(M_i)$ local. Then $M$ has the unique decomposition property.

*Proof.* Let
$$M = \bigoplus_{i=1}^m M_i = \bigoplus_{i=1}^n M_i'.$$
We prove by induction on $m$. If $m = 1$, then $M$ is indecomposable. Then we must have $n = 1$ as well, and the result follows.

For $m > 1$, we consider the maps
$$\alpha_i : M_i' \hookrightarrow M \twoheadrightarrow M_1$$
$$\beta_i : M_1 \hookrightarrow M \twoheadrightarrow M_i'$$
We observe that
$$\mathrm{id}_{M_1} = \sum_{i=1}^n \alpha_i \circ \beta_i : M_1 \to M_1.$$
Since $\mathrm{End}_A(M_1)$ is local, we know some $\alpha_i \circ \beta_i$ must be invertible, i.e. a unit, as they cannot all lie in the Jacobson radical. We may wlog assume $\alpha_1 \circ \beta_1$ is a unit. If this is the case, then both $\alpha_1$ and $\beta_1$ have to be invertible. So $M_1 \cong M_1'$.

Consider the map $\mathrm{id} - \theta = \phi$, where
$$\theta : M \twoheadrightarrow M_1 \xrightarrow{\alpha_1^{-1}} M_1' \hookrightarrow M \twoheadrightarrow \textstyle\bigoplus_{i=2}^m M_i \hookrightarrow M.$$
Then $\phi(M_1') = M_1$. So $\phi|_{M_1'}$ looks like $\alpha_1$. Also
$$\phi\left(\bigoplus_{i=2}^m M_i\right) = \bigoplus_{i=2}^m M_i,$$
So $\phi|_{\bigoplus_{i=2}^m M_i}$ looks like the identity map. So in particular, we see that $\phi$ is surjective. However, if $\phi(x) = 0$, this says $x = \theta(x)$, So
$$x \in \bigoplus_{i=2}^m M_i.$$
But then $\theta(x) = 0$. Thus $x = 0$. Thus $\phi$ is an automorphism of $m$ with $\phi(M_1') = \phi(M_1)$. So this gives an isomorphism between
$$\bigoplus_{i=2}^m M_i = \frac{M}{M_1} \cong \frac{M}{M_1} \cong \bigoplus_{i=2}^n M_i',$$
and so we are done by induction. □

**Lemma** (Fitting)**.** Suppose $M$ is a module with both the ACC and DCC on submodules, and let $f \in \mathrm{End}_A(M)$. Then for large enough $n$, we have
$$M = \operatorname{im} f^n \oplus \ker f^n.$$

*Proof.* By ACC and DCC, we may choose $n$ large enough so that
$$f^n : f^n(M) \to f^{2n}(M)$$

is an isomorphism, as if we keep iterating $f$, the image is a descending chain and the kernel is an ascending chain, and these have to terminate.

If $m \in M$, then we can write

$$f^n(m) = f^{2n}(m_1)$$

for some $m_1$. Then

$$m = f^n(m_1) + (m - f^n(m_1)) \in \operatorname{im} f^n + \ker f^n,$$

and also

$$\operatorname{im} f^n \cap \ker f^n = \ker(f^n : f^n(M) \to f^{2n}(M)) = 0.$$

So done. □

**Lemma.** Suppose $M$ is an indecomposable module satisfying ACC and DCC on submodules. Then $B = \operatorname{End}_A(M)$ is local.

*Proof.* Choose a maximal left ideal of $B$, say $I$. It's enough to show that if $x \notin I$, then $x$ is left invertible. By maximality of $I$, we know $B = Bx + I$. We write

$$1 = \lambda x + y,$$

for some $\lambda \in B$ and $y \in I$. Since $y \in I$, it has no left inverse. So it is not an isomorphism. By Fitting's lemma and the indecomposability of $M$, we see that $y^m = 0$ for some $m$. Thus

$$(1 + y + y^2 + \cdots + y^{m-1})\lambda x = (1 + y + \cdots + y^{m-1})(1 - y) = 1.$$

So $x$ is left invertible. □

**Corollary.** Let $A$ be a left Artinian algebra. Then $A$ has the unique decomposition property.

*Proof.* We know $A$ satisfies the ACC and DCC condition. So ${}_AA$ is a finite direct sum of indecomposables. □

**Proposition.** Let $N$ be a nilpotent ideal in $A$, and let $f$ be an idempotent of $A/N \equiv \bar{A}$. Then there is an idempotent $e \in A$ with $f = \bar{e}$.

*Proof.* We consider the quotients $A/N^i$ for $i \geq 1$. We will lift the idempotents successively as we increase $i$, and since $N$ is nilpotent, repeating this process will eventually land us in $A$.

Suppose we have found an idempotent $f_{i-1} \in A/N^{i-1}$ with $\bar{f}_{i-1} = f$. We want to find $f_i \in A/N^i$ such that $\bar{f}_i = f$.

For $i > 1$, we let $x$ be an element of $A/N^i$ with image $f_{i-1}$ in $A/N^{i-1}$. Then since $x^2 - x$ vansishes in $A/N^{i-1}$, we know $x^2 - x \in N^{i-1}/N^i$. Then in particular,

$$(x^2 - x)^2 = 0 \in A/N^i. \tag{$\dagger$}$$

We let

$$f_i = 3x^2 - 2x^3.$$

Then by a direct computation using (†), we find $f_i^2 = f_i$, and $f_i$ has image $3f_{i-1} - 2f_{i-1} = f_{i-1}$ in $A/N^{i-1}$ (alternatively, in characteristic $p$, we can use $f_i = x^p$). Since $N^k = 0$ for some $k$, this process gives us what we want. □

**Corollary.** Let $N$ be a nilpotent ideal of $A$. Let

$$\bar{1} = f_1 + \cdots + f_r$$

with $\{f_i\}$ orthogonal primitive idempotents in $A/N$. Then we can write

$$1 = e_1 + \cdots + e_r,$$

with $\{e_i\}$ orthogonal primitive idempotents in $A$, and $\bar{e}_i = f_i$.

*Proof.* We define a sequence $e_i' \in A$ inductively. We set

$$e_1' = 1.$$

Then for each $i > 1$, we pick $e_i'$ a lift of $f_i + \cdots + f_t \in e_{i-1}' A e_{i-1}'$, since by inductive hypothesis we know that $f_i + \cdots + f_t \in e_{i-1}' A e_{i-1}'/N$. Then

$$e_i' e_{i+1}' = e_{i+1}' = e_{i+1}' e_i'.$$

We let

$$e_i = e_i' - e_{i+1}'.$$

Then

$$\bar{e}_i = f_i.$$

Also, if $j > i$, then

$$e_j = e_{i+1}' e_j e_{i+1}',$$

and so

$$e_i e_j = (e_i' - e_{i+1}') e_{i+1}' e_j e_{i+1}' = 0.$$

Similarly $e_j e_i = 0$. □

**Lemma.** Let $P$ be an indecomposable projective, and $M$ an $A$-module. Then $\mathrm{Hom}(P, M) \neq 0$ iff $P/PJ(A)$ is a composition factor of $M$.

*Proof.* We have proven $\Rightarrow$. Conversely, suppose there is a non-zero map $f : P \to M$. Then it factors as

$$S = \frac{P}{PJ(A)} \to \frac{\operatorname{im} f}{(\operatorname{im} f)J(A)}.$$

Now we cannot have $\operatorname{im} f = (\operatorname{im} f)J(A)$, or else we have $\operatorname{im} f = (\operatorname{im} f)J(A)^n = 0$ for sufficiently large $n$ since $J(A)$ is nilpotent. So this map must be injective, hence an isomorphism. So this exhibits $S$ as a composition factor of $M$. □

**Theorem.** Indecomposable projectives $P_1$ and $P_2$ are in the same block if and only if they lie in the same connected component of the graph.

*Proof.* It is clear that $P_1$ and $P_2$ are in the same connected component, then they are in the same block.

Conversely, consider a connected component $X$, and consider

$$I = \bigoplus_{P \in X} P.$$

We show that this is in fact a left ideal, hence an ideal. Consider any $x \in A$. Then for each $P \in X$, left-multiplication gives a map $P \to A$, and if we decompose

$$A = \bigoplus P_i,$$

then this can be expressed as a sum of maps $f_i : P \to P_i$. Now such a map can be non-zero only if $P$ is a composition factor of $P_i$. So if $f_i \neq 0$, then $P_i \in X$. So left-multiplication by $x$ maps $I$ to itself, and it follows that $I$ is an ideal. □

## 1.5 $K_0$

# 2 Noetherian algebras

## 2.1 Noetherian algebras

**Theorem** (Hilbert basis theorem)**.** If $A$ is Noetherian, then $A[X]$ is Noetherian.

**Theorem.** Let $A$ be left Noetherian. Then $A[[X]]$ is Noetherian.

*Proof.* Let $I$ be a left ideal of $A[[X]]$. We'll show that if $A$ is left Noetherian, then $I$ is finitely generated. Let

$$J_r = \{a : \text{there exists an element of } I \text{ of the form } aX^r + \text{higher degree terms}\}.$$

We note that $J_r$ is a left ideal of $A$, and also note that

$$J_0 \leq J_1 \leq J_2 \leq J_3 \leq \cdots,$$

as we can always multiply by $X$. Since $A$ is left Noetherian, this chain terminates at $J_N$ for some $N$. Also, $J_0, J_1, J_2, \cdots, J_N$ are all finitely generated left ideals. We suppose $a_{i1}, \cdots, a_{is_i}$ generates $J_i$ for $i = 1, \cdots, N$. These correspond to elements

$$f_{ij}(X) = a_{ij}X^j + \text{higher odder terms} \in I.$$

We show that this finite collection generates $I$ as a left ideal. Take $f(X) \in I$, and suppose it looks like

$$b_n X^n + \text{higher terms},$$

with $b_n \neq 0$.

Suppose $n < N$. Then $b_n \in J^n$, and so we can write

$$b_n = \sum c_{nj} a_{nj}.$$

So

$$f(X) - \sum c_{nj} f_{nj}(X) \in I$$

has zero coefficient for $X^n$, and all other terms are of higher degree.

Repeating the process, we may thus wlog $n \geq N$. We get $f(X)$ of the form $d_N X^N +$ higher degree terms. The same process gives

$$f(X) - \sum c_{Nj} f_{Nj}(X)$$

with terms of degree $N + 1$ or higher. We can repeat this yet again, using the fact $J_N = J_{N+1}$, so we obtain

$$f(X) - \sum c_{Nj} f_{Nj}(x) - \sum d_{N+1,j} X f_{Nj}(X) + \cdots.$$

So we find

$$f(X) = \sum e_j(X) f_{Nj}(X)$$

for some $e_j(X)$. So $f$ is in the left ideal generated by our list, and hence so is $f$. $\square$

**Lemma.** Let $A$ be a positively filtered algebra. If $\operatorname{gr} A$ is Noetherian, then $A$ is left Noetherian.

*Proof.* Given a left ideal $I$ of $A$, we can form
$$\operatorname{gr} I = \bigoplus \frac{I \cap A_i}{I \cap A_{i-1}},$$
where $I$ is filtered by $\{I \cap A_i\}$. By the isomorphism theorem, we know
$$\frac{I \cap A_i}{I \cap A_{i-1}} \cong \frac{I \cap A_i + A_{i-1}}{A_{i-1}} \subseteq \frac{A_i}{A_{i-1}}.$$
Then $\operatorname{gr} I$ is a left graded ideal of $\operatorname{gr} A$.

Now suppose we have a strictly ascending chain
$$I_1 < I_2 < \cdots$$
of left ideals. Since we have a positive filtration, for some $A_i$, we have $I_1 \cap A_i \subsetneq I_2 \cap A_i$ and $I_1 \cap A_{i-1} = I_2 \cap A_{i-1}$. Thus
$$\operatorname{gr} I_1 \subsetneq \operatorname{gr} I_2 \subsetneq \operatorname{gr} I_3 \subsetneq \cdots.$$
This is a contradiction since $\operatorname{gr} A$ is Noetherian. So $A$ must be Noetherian. □

**Corollary.** $A_n(k)$ and $\mathcal{U}(\mathfrak{g})$ are left/right Noetherian.

*Proof.* $\operatorname{gr} A_n(k)$ and $\operatorname{gr} \mathcal{U}(\mathfrak{g})$ are commutative and finitely generated algebras. □

## 2.2 More on $A_n(k)$ and $\mathcal{U}(\mathfrak{g})$

**Lemma.** Suppose $\operatorname{char} k = 0$. Then $A_n(k)$ has no non-zero modules that are finite-dimensional $k$-vector spaces.

*Proof.* Suppose $M$ is a finite-dimensional module. Then we've got an algebra homomorphism $\theta : A_n(k) \to \operatorname{End}_k(M) \cong M_m(k)$, where $m = \dim_k M$.

In $A_n(k)$, we have
$$Y_1 X_1 - X_1 Y_1 = 1.$$
Applying the trace map, we know
$$\operatorname{tr}(\theta(Y_1)\theta(X_1) - \theta(X_1)\theta(Y_1)) = \operatorname{tr} I = m.$$
But since the trace is cyclic, the left hand side vanishes. So $m = 0$. So $M$ is trivial. □

**Theorem** (Hilbert-Serre theorem)**.** The Poincaré series $P(V, t)$ of a finitely-generated graded module
$$V = \bigoplus_{i=0}^\infty V_i$$
over a finitely-generated generated commutative algebra
$$S = \bigoplus_{i=0}^\infty S_i$$
with homogeneous generating set $x_1, \cdots, x_m$ is a rational function of the form
$$\frac{f(t)}{\prod (1 - t^{k_i})},$$
where $f(t) \in \mathbb{Z}[t]$ and $k_i$ is the degree of the generator $x_i$.

*Proof.* We induct on the number $m$ of generators. If $m = 0$, then $S = S_0 = k$, and $V$ is therefore a finite-dimensional $k$-vector space. So $P(V, t)$ is a polynomial.

Now suppose $m > 0$. We assume the theorem is true for $< m$ generators. Consider multiplication by $x_m$. This gives a map

$$V_i \xrightarrow{x_m} V_{i+k_m} ,$$

and we have an exact sequence

$$0 \longrightarrow K_i \longrightarrow V_i \xrightarrow{x_m} V_{i+k_m} \longrightarrow L_{i+k_m} \to 0, \qquad (*)$$

where

$$K = \bigoplus K_i = \ker(x_m : V \to V)$$

and

$$L = \bigoplus L_{i+k_m} = \operatorname{coker}(x_m : V \to V).$$

Then $K$ is a graded submodule of $V$ and hence is a finitely-generated $S$-module, using the fact that $S$ is Noetherian. Also, $L = V/x_mV$ is a quotient of $V$, and it is thus also finitely-generated.

Now both $K$ and $L$ are annihilated by $x_m$. So they may be regarded as $S_0[x_1, \cdots, x_{m-1}]$-modules. Applying $\dim_k$ to $(*)$, we know

$$\dim_k(K_i) - \dim_k(V_i) + \dim(V_{i+k_m}) - \dim(L_{i+k_m}) = 0.$$

We multiply by $t^{i+k_m}$, and sum over $i$ to get

$$t^{k_m}P(K, t) - t^{k_m}P(V, t) + P(V, t) - P(L, t) = g(t),$$

where $g(t)$ is a polynomial with integral coefficients arising from consideration of the first few terms.

We now apply the induction hypothesis to $K$ and $L$, and we are done. $\square$

**Corollary.** If each $k_1, \cdots, k_m = 1$, i.e. $S$ is generated by $S_0 = k$ and homogeneous elements $x_1, \cdots, x_m$ of degree 1, then for large enough $i$, then $\dim V_i = \phi(i)$ for some polynomial $\phi(t) \in \mathbb{Q}[t]$ of $d - 1$, where $d$ is the order of the pole of $P(V, t)$ at $t = 1$. Moreover,

$$\sum_{j=0}^{i} \dim V_j = \chi(i),$$

where $\chi(t) \in \mathbb{Q}[t]$ of degree $d$.

*Proof.* From the theorem, we know that

$$P(V, t) = \frac{f(t)}{(1 - t)^d},$$

for some $d$ with $f(1) \neq 0$, $f \in \mathbb{Z}[t]$. But

$$(1 - t)^{-1} = 1 + t + t^2 + \cdots$$

By differentiating, we get an expression

$$(1-t)^{-d} = \sum \binom{d+i-1}{d-1} t^i.$$

If

$$f(t) = a_0 + a_1 t + \cdots + a_s t^s,$$

then we get

$$\dim V_i = a_0 \binom{d+i-1}{d-1} + a_1 \binom{d+i-2}{d-1} + \cdots + a_s \binom{d+i-s-1}{d-1},$$

where we set $\binom{r}{d-1} = 0$ if $r < d-1$, and this expression can be rearranged to give $\phi(i)$ for a polynomial $\phi(t) \in \mathbb{Q}[t]$, valid for $i - s > 0$. In fact, we have

$$\phi(t) = \frac{f(1)}{(d-1)!} t^{d-1} + \text{lower degree term}.$$

Since $f(1) \neq 0$, this has degree $d-1$.

This implies that

$$\sum_{j=0}^{i} \dim V_j$$

is a polynomial in $\mathbb{Q}[t]$ of degree $d$. □

**Lemma.** Let $M$ be a finitely-generated $A_n$-module. Then $d(M) \leq 2n$.

*Proof.* Take generators $m_1, \cdots, m_s$ of $M$. Then there is a surjective filtered module homomorphism

$$\begin{aligned} A_n \oplus \cdots \oplus A_n &\longrightarrow M \\ (a_1, \cdots, a_s) &\longmapsto \sum a_i m_i \end{aligned}$$

It is easy to see that quotients can only reduce dimension, so

$$\text{GK-dim}(M) \leq d(A_n \oplus \cdots \oplus A_n).$$

But

$$\chi_{A_n \oplus \cdots \oplus A_n} = s\chi_{A_n}$$

has degree $2n$. □

**Theorem** (Bernstein's inequality)**.** Let $M$ be a non-zero finitely-generated $A_n(k)$-module, and $\operatorname{char} k = 0$. Then

$$d(M) \geq n.$$

*Proof.* Take a generating set and form the canonical filtrations $\{A_i\}$ of $A_n(k)$ and $\{M_i\}$ of $M$. We let $\chi(t)$ be the Samuel polynomial. Then for large enough $i$, we have

$$\chi(i) = \dim M_i.$$

We claim that

$$\dim A_i \leq \dim \mathrm{Hom}_k(M_i, M_{2i}) = \dim M_i \times \dim M_{2i}.$$

Assuming this, for large enough $i$, we have

$$\dim A_i \leq \chi(i)\chi(2i).$$

But we know

$$\dim A_i = \binom{i+2}{2n},$$

which is a polynomial of degree $2n$. But $\chi(t)\chi(2t)$ is a polynomial of degree $2d(M)$. So we get that

$$n \leq d(M).$$

So it remains to prove the claim. It suffices to prove that the natural map

$$A_i \to \mathrm{Hom}_k(M_i, M_{2i}),$$

given by multiplication is injective.

So we want to show that if $a \in A_i \neq 0$, then $aM_i \neq 0$. We prove this by induction on $i$. When $i = 0$, then $A_0 = k$, and $M_0$ is a finite-dimensional $k$-vector space. Then the result is obvious.

If $i > 0$, we suppose the result is true for smaller $i$. We let $a \in A_i$ is non-zero. If $aM_i = 0$, then certainly $a \notin k$. We express

$$a = \sum c_{\boldsymbol{\alpha}\boldsymbol{\beta}} X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_n^{\beta_n},$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)$, and $c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in k$.

If possible, pick a $j$ such that $c_{\boldsymbol{\alpha},\boldsymbol{\alpha}} \neq 0$ for some $\boldsymbol{\alpha}$ with $\alpha_j \neq 0$ (this happens when there is an $X$ involved). Then

$$[Y_j, a] = \sum \alpha_j c_{\boldsymbol{\alpha},\boldsymbol{\beta}} X_1^{\alpha_1} \cdots X_j^{\alpha_j - 1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_n^{\beta_n},$$

and this is non-zero, and lives in $A_{i-1}$.

If $aM_i = 0$, then certainly $aM_{i-1} = 0$. Hence

$$[Y_j, a]M_{i-1} = (Y_j a - aY_j)M_{i-1} = 0,$$

using the fact that $Y_j M_{i-1} \subseteq M_i$. This is a contradiction.

If $a$ only has $Y$'s involved, then we do something similar using $[X_j, a]$. $\square$

## 2.3 Injective modules and Goldie's theorem

**Theorem** (Goldie's theorem)**.** Let $A$ be a right Noetherian algebra with no non-zero ideals all of whose elements are nilpotent. Then $A$ embeds in a finite direct sum of matrix algebras over division algebras.

**Lemma.** Every direct summand of an injective module is injective, and direct products of injectives is injective.

*Proof.* Same as proof for projective modules. $\square$

**Lemma.** Every $A$-module may be embedded in an injective module.

*Proof.* Let $M$ be a right $A$-module. Then $\mathrm{Hom}_k(A, M)$ is a right $A$-module via

$$(fa)(x) = f(ax).$$

We claim that $\mathrm{Hom}_k(A, M)$ is an injective module. Suppose we have

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "\theta"] \arrow[d, "\phi"] & N_1 \\ & \mathrm{Hom}_k(A, M) \end{tikzcd}$$

We consider the $k$-module diagram

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "\theta"] \arrow[d, "\alpha"] & N_1 \arrow[dl, dashed, "\beta"] \\ & M \end{tikzcd}$$

where $\alpha(m_1) = \phi(m_1)(1)$. Since $M$ is injective as a $k$-module, we can find the $\beta$ such that $\alpha = \beta\theta$. We define $\psi : N_1 \to \mathrm{Hom}_k(A, M)$ by

$$\psi(n_1)(x) = \beta(n_1 x).$$

It is straightforward to check that this does the trick. Also, we have an embedding $M \hookrightarrow \mathrm{Hom}_k(A, M)$ by $m \mapsto (\phi_n : x \mapsto mx)$. □

**Lemma.** An $A$-module is injective iff it is a direct summand of every extension of itself.

*Proof.* Suppose $E$ is injective and $E'$ is an extension of $E$. Then we can form the diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d, "\mathrm{id}"] & E' \arrow[dl, dashed, "\psi"] \\ & E \end{tikzcd},$$

and then by injectivity, we can find $\psi$. So

$$E' = E \oplus \ker \psi.$$

Conversely, suppose $E$ is a direct summand of every extension. But by the previous lemma, we can embed $E$ in an injective $E'$. This implies that $E$ is a direct summand of $E'$, and hence injective. □

**Lemma.** An essential extension of an essential extension is essential.

*Proof.* Suppose $M < E < F$ are essential extensions. Then given $N \leq F$, we know $N \cap E \neq \{0\}$, and this is a submodule of $E$. So $(N \cap E) \cap M = N \cap M \neq 0$. So $F$ is an essential extension of $M$. □

**Lemma.** A maximal essential extension is an injective module.

*Proof.* Let $E$ be a maximal essential extension of $M$, and consider any embedding $E \hookrightarrow F$. We shall show that $E$ is a direct summand of $F$. Let $S$ be the set of all non-zero submodules $V$ of $F$ with $V \cap E = \{0\}$. We apply Zorn's lemma to get a maximal such module, say $V_1$.

Then $E$ embeds into $F/V_1$ as an essential submodule. By transitivity of essential extensions, $F/V_1$ is an essential extension of $M$, but $E$ is maximal. So $E \cong F/V_1$. In other words,
$$F = E \oplus V_1. \qquad \square$$

**Proposition.** Let $M$ be an $A$-module, with an inclusion $M \hookrightarrow I$ into an injective module. Then this extends to an inclusion $E(M) \hookrightarrow I$.

*Proof.* By injectivity, we can fill in the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, hook] \arrow[d, hook] & E(M) \arrow[dl, dashed, "\psi"] \\ & I & \end{tikzcd} .$$

We know $\psi$ restricts to the identity on $M$. So $\ker \psi \cap M = \{0\}$. By Since $E(M)$ is essential, we must have $\ker \psi = 0$. So $E(M)$ embeds into $I$. $\square$

**Proposition.** Suppose $E$ is an injective essential extension of $M$. Then $E \cong E(M)$. In particular, any two injective hulls are isomorphic.

*Proof.* By the previous lemma, $E(M)$ embeds into $E$. But $E(M)$ is a maximal essential extension. So this forces $E = E(M)$. $\square$

**Proposition.**
$$E(M_1 \oplus M_2) = E(M_1) \oplus E(M_2).$$

*Proof.* We know that $E(M_1) \oplus E(M_2)$ is also injective (since finite direct sums are the same as direct products), and also $M_1 \oplus M_2$ embeds in $E(M_1) \oplus E(M_2)$. So it suffices to prove this extension is essential.

Let $V \leq E(M_1) \oplus E(M_2)$. Then either $V/E(M_1) \neq 0$ or $V/E(M_2) \neq 0$.

We wlog it is the latter. Note that we can naturally view
$$\frac{V}{E(M_2)} \leq \frac{E(M_1) \oplus E(M_2)}{E(M_2)} \cong E(M_1).$$

Since $M_1 \subseteq E(M_1)$ is essential, we know
$$M_1 \cap (V/E(M_2)) \neq 0.$$

So there is some $m_1 + m_2 \in V$ such that $m_2 \in E(M_2)$ and $m_1 \in M_1$. Now consider
$$\{m \in E(M_2) : am_1 + m \in V \text{ for some } a \in A\}.$$

This is a non-empty submodule of $E(M_2)$, and so contains an element of $M_2$, say $n$. Then we know $am_1 + n \in V \cap (M_1 \oplus M_2)$, and we are done. $\square$

**Lemma.** $V$ is uniform iff $E(V)$ is indecomposable.

*Proof.* Suppose $E(V) = A \oplus B$, with $A, B$ non-zero. Then $V \cap A \neq \{0\}$ and $V \cap B \neq \{0\}$ since the extension is essential. So we have two non-zero submodules of $V$ that intersect trivially.

Conversely, suppose $V$ is not uniform, and let $V_1, V_2$ be non-zero submodules that intersect trivially. By Zorn's lemma, we suppose these are maximal submodules that intersect trivially. We claim

$$E(V_1) \oplus E(V_2) = E(V_1 \oplus V_2) = E(V)$$

To prove this, it suffices to show that $V$ is an essential extension of $V_1 \oplus V_2$, so that $E(V)$ is an injective hull of $V_1 \oplus V_2$.

Let $W \leq V$ be non-zero. If $W \cap (V_1 \oplus V_2) = 0$, then $V_1 \oplus (V_2 \oplus W)$ is a larger pair of submodules with trivial intersection, which is not possible. So we are done. $\square$

**Lemma.** Let $A$ be a filtered algebra, which is exhaustive and separated. Then if $\operatorname{gr} A$ is a domain, then so is $A$.

*Proof.* Let $x \in A_i \setminus A_{i-1}$, and $y \in A_j \setminus A_{j-1}$. We can find such $i, j$ for any elements $x, y \in A$ because the filtration is exhaustive and separated. Then we have

$$\begin{aligned} \bar{x} &= x + A_{i-1} \neq 0 \in A_i/A_{i-1} \\ \bar{y} &= y + A_{j-1} \neq 0 \in A_j/A_{j-1}. \end{aligned}$$

If $\operatorname{gr} A$ is a domain, then we deduce $\bar{x}\bar{y} \neq 0$. So we deduce that $xy \notin A_{i+j-1}$. In particular, $xy \neq 0$. $\square$

**Corollary.** $A_n(k)$ and $\mathcal{U}(\mathfrak{g})$ are domains.

**Lemma.** Let $A$ be a right Noetherian domain. Then $A_A$ is uniform, i.e. $E(A_A)$ is indecomposable.

*Proof.* Suppose not, and so there are $xA$ and $yA$ non-zero such that $xA \cap yA = \{0\}$. So $xA \oplus yA$ is a direct sum.

But $A$ is a domain and so $yA \cong A$ as a right $A$-module. Thus $yxA \oplus yyA$ is a direct sum living inside $yA$. Further decomposing $yyA$, we find that

$$xA \oplus yxA \oplus y^2xA \oplus \cdots \oplus y^n xA$$

is a direct sum of non-zero submodules. But this is an infinite strictly ascending chain as $n \to \infty$, which is a contradiction. $\square$

**Lemma.** Let $E$ be an indecomposable injective right module. Then $\operatorname{End}_A(E)$ is a local algebra, with the unique maximal ideal given by

$$I = \{f \in \operatorname{End}(E) : \ker f \text{ is essential}\}.$$

*Proof.* Let $f : E \to E$ and $\ker f = \{0\}$. Then $f(E)$ is an injective module, and so is a direct summand of $E$. But $E$ is indecomposable. So $f$ is surjective. So it is an isomorphism, and hence invertible. So it remains to show that

$$I = \{f \in \operatorname{End}(E) : \ker f \text{ is essential}\}$$

is an ideal.

If $\ker f$ and $\ker g$ are essential, then $\ker(f + g) \geq \ker f \cap \ker g$, and the intersection of essential submodules is essential. So $\ker(f + g)$ is also essential.

Also, if $\ker g$ is essential, and $f$ is arbitrary, then $\ker(f \circ g) \geq \ker g$, and is hence also essential. So $I$ is a maximal left ideal. □

**Lemma.** Let $M$ be a non-zero Noetherian module. Then $M$ is an essential extension of a direct sum of uniform submodules $N_1, \cdots, N_r$. Thus

$$E(M) \cong E(N_1) \oplus \cdots E(N_r)$$

is a direct sum of finitely many indecomposables.

This decomposition is unique up to re-ordering (and isomorphism).

*Proof.* We first show any non-zero Noetherian module contains a uniform one. Suppose not, and $M$ is in particular not uniform. So it contains non-zero $V_1, V_2'$ with $V_1 \cap V_2' = 0$. But $V_2'$ is not uniform by assumption. So it contains non-zero $V_2$ and $V_3'$ with zero intersection. We keep on repeating. Then we get

$$V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

is a strictly ascending chain of submodules of $M$, which is a contradiction.

Now for non-zero Noetherian $M$, pick $N_1$ uniform in $M$. Either $N_1$ is essential in $M$, and we're done, or there is some $N_2'$ non-zero with $N_1 \cap N_2' = 0$. We pick $N_2$ uniform in $N_2'$. Then either $N_1 \oplus N_2$ is essential, or...

And we are done since $M$ is Noetherian. Taking injective hulls, we get

$$E(M) = E(N_1) \oplus \cdots \oplus E(N_r),$$

and we are done by Krull–Schmidt and the previous lemma. □

**Lemma.** Let $E_1, \cdots, E_r$ be indecomposable injectives. Put $E = E_1 \oplus \cdots \oplus E_r$. Let $I = \{f \in \mathrm{End}_A(E) : \ker f \text{ is essential}\}$. This is an ideal, and then

$$\mathrm{End}_A(E)/I \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_s}(D_s)$$

for some division algebras $D_i$.

*Proof.* We write the decomposition instead as

$$E = E_1^{n_1} \oplus \cdots \oplus E_r^{n_r}.$$

Then as in basic linear algebra, we know elements of $\mathrm{End}(E)$ can be written as an $r \times r$ matrix whose $(i, j)$th entry is an element of $\mathrm{Hom}(E_i^{n_i}, E_j^{n_j})$.

Now note that if $E_i \not\cong E_j$, then the kernel of a map $E_i \to E_j$ is essential in $E_i$. So quotienting out by $I$ kills all of these "off-diagonal" entries.

Also $\mathrm{Hom}(E_i^{n_i}, E_i^{n_i}) = M_{n_i}(\mathrm{End}(E_i))$, and so quotienting out by $I$ gives $M_{n_i}(\mathrm{End}(E_i)/\{\text{essential kernel}\}) \cong M_{n_i}(D_i)$, where

$$D_i \cong \frac{\mathrm{End}(E_i)}{\text{essential kernel}},$$

which we know is a division algebra since $I$ is a maximal ideal. □

**Lemma.** If $A$ is a right Noetherian algebra, then any $f : A_A \to A_A$ with $\ker f$ essential in $A_A$ is nilpotent.

*Proof.* Consider
$$0 < \ker f \leq \ker f^2 \leq \cdots .$$
Suppose $f$ is not nilpotent. We claim that this is a strictly increasing chain. Indeed, for all $n$, we have $f^n(A_A) \neq 0$. Since $\ker f$ is essential, we know
$$f^n(A_A) \cap \ker f \neq \{0\}.$$
This forces $\ker f^{n+1} > \ker f^n$, which is a contradiction. □

**Theorem** (Goldie's theorem)**.** Let $A$ be a right Noetherian algebra with no non-zero ideals all of whose elements are nilpotent. Then $A$ embeds in a finite direct sum of matrix algebras over division algebras.

*Proof.* As usual, we have a map
$$\begin{aligned} A &\longrightarrow \mathrm{End}_A(A_A) \\ x &\longmapsto \text{left multiplication by } x \end{aligned}$$
For a map $A_A \to A_A$, it lifts to a map $E(A_A) \to E(A_A)$ by injectivity:
$$\begin{tikzcd} 0 \arrow[r] & A_A \arrow[r, "\theta"] \arrow[d, "f"] & E(A_A) \arrow[d, dashed, "f'"] \\ & A_A \arrow[r, "\theta"] & E(A_A) \end{tikzcd}$$
We can complete the diagram to give a map $f' : E(A_A) \to E(A_A)$, which restricts to $f$ on $A_A$. This is not necessarily unique. However, if we have two lifts $f'$ and $f''$, then the difference $f' - f''$ has $A_A$ in the kernel, and hence has an essential kernel. So it lies in $I$. Thus, if we compose maps
$$A_A \longrightarrow \mathrm{End}_A(A_A) \longrightarrow \mathrm{End}(E(A_A))/I \ .$$
The kernel of this consists of $A$ which when multiplying on the left has essential kernel. This is an ideal all of whose elements is nilpotent. By assumption, any such ideal vanishes. So we have an embedding of $A$ in $\mathrm{End}(E(A_A))/I$, which we know to be a direct sum of matrix algebras over division rings. □

# 3 Hochschild homology and cohomology

## 3.1 Introduction

## 3.2 Cohomology

**Lemma.** Let $M$ be an injective bimodule. Then $HH^n(A, M) = 0$ for all $n \geq 1$.

*Proof.* $\mathrm{Hom}_{A\text{-}A}(\,\cdot\,, M)$ is exact. □

**Lemma.** If ${}_AA_A$ is a projective bimodule, then $HH^n(A, M) = 0$ for all $M$ and all $n \geq 1$.

*Proof.* If ${}_AA_A$ is projective, then all $A^{\otimes n}$ are projective. At each degree $n$, we can split up the Hochschild chain complex as the short exact sequence

$$0 \longrightarrow \frac{A^{\otimes(n+3)}}{\ker d_n} \xhookrightarrow{d_n} A^{\otimes(n+2)} \xrightarrow{d_{n-1}} \operatorname{im} d_{n-1} \longrightarrow 0$$

The $\operatorname{im} d$ is a submodule of $A^{\otimes(n+1)}$, and is hence projective. So we have

$$A^{\otimes(n+2)} \cong \frac{A^{\otimes(n+3)}}{\ker d_n} \oplus \operatorname{im} d_{n-1},$$

and we can write the Hochschild chain complex at $n$ as

$$\begin{array}{ccccc} \ker d_n \oplus \dfrac{A^{\otimes(n+3)}}{\ker d_n} & \xrightarrow{d_n} & \dfrac{A^{\otimes(n+3)}}{\ker d_n} \oplus \operatorname{im} d_{n-1} & \xrightarrow{d_{n-1}} & \dfrac{A^{\otimes(n+1)}}{\operatorname{im} d_{n-1}} \oplus \operatorname{im} d_{n-1} \\ (a, b) & \longmapsto & (b, 0) & & \\ & & (c, d) & \longmapsto & (0, d) \end{array}$$

Now $\mathrm{Hom}(\,\cdot\,, M)$ certainly preserves the exactness of this, and so the Hochschild cochain complex is also exact. So we have $HH^n(A, M) = 0$ for $n \geq 1$. □

**Lemma.** If $\operatorname{Dim} A = 0$, then $A$ is separable.

*Proof.* Note that there is a short exact sequence

$$0 \longrightarrow \ker \mu \longrightarrow A \otimes A \xrightarrow{\mu} A \longrightarrow 0$$

If we can show this splits, then $A$ is a direct summand of $A \otimes A$. To do so, we need to find a map $A \otimes A \to \ker \mu$ that restricts to the identity on $\ker \mu$.

To do so, we look at the first few terms of the Hochschild chain complex

$$\cdots \xrightarrow{d} \operatorname{im} d \oplus \ker \mu \longrightarrow A \otimes A \xrightarrow{\mu} A \longrightarrow 0\ .$$

By assumption, for any $M$, applying $\mathrm{Hom}_{A\text{-}A}(\,\cdot\,, M)$ to the chain complex gives an exact sequence. Omitting the annoying ${}_{A\text{-}A}$ subscript, this sequence looks like

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(A, M) &\xrightarrow{\mu^*} \mathrm{Hom}(A \otimes A, M) \\ &\xrightarrow{(*)} \mathrm{Hom}(\ker \mu, M) \oplus \mathrm{Hom}(\operatorname{im} d, M) \xrightarrow{d^*} \cdots \end{aligned}$$

Now $d^*$ sends $\mathrm{Hom}(\ker \mu, M)$ to zero. So $\mathrm{Hom}(\ker \mu, M)$ must be in the image of $(*)$. So the map

$$\mathrm{Hom}(A \otimes A, M) \longrightarrow \mathrm{Hom}(\ker \mu, M)$$

must be surjective. This is true for any $M$. In particular, we can pick $M = \ker \mu$. Then the identity map $\mathrm{id}_{\ker \mu}$ lifts to a map $A \otimes A \to \ker \mu$ whose restriction to $\ker \mu$ is the identity. So we are done. □

**Proposition.** We have

$$HH^0(A, M) = \{m \in M : am - ma = 0 \text{ for all } a \in A\}.$$

In particular, $HH^0(A, A)$ is the center of $A$.

**Proposition.**

$$\ker \delta_1 = \{f \in \mathrm{Hom}_k(A, M) : f(a_1 a_2) = a_1 f(a_2) + f(a_1) a_2\}.$$

These are the *derivations* from $A$ to $M$. We write this as $\mathrm{Der}(A, M)$.

On the other hand,

$$\operatorname{im} \delta_0 = \{f \in \mathrm{Hom}_k(A, M) : f(a) = am - ma \text{ for some } m \in M\}.$$

These are called the *inner derivations* from $A$ to $M$. So

$$HH^1(A, M) = \frac{\mathrm{Der}(A, M)}{\mathrm{InnDer}(A, M)}.$$

Setting $A = M$, we get the derivations and inner derivations of $A$.

**Lemma.** We have

$$\mathrm{Der}_k(A, M) \cong \{\text{algebra complements to } M \text{ in } A \ltimes M \text{ isomorphic to } A\}.$$

*Proof.* A complement to $M$ is an embedded copy of $A$ in $A \ltimes M$,

$$\begin{aligned} A &\longrightarrow A \ltimes M \\ a &\longmapsto (a, D_a) \end{aligned}$$

The function $A \to M$ given by $a \mapsto D_a$ is a derivation, since under the embedding, we have

$$ab \mapsto (ab, aD_b + D_a b).$$

Conversely, a derivation $f : A \to M$ gives an embedding of $A$ in $A \ltimes M$ given by $a \mapsto (a, f(a))$. □

**Lemma.** We have

$$\mathrm{Der}(A, M) \cong \left\{ \begin{array}{c} \text{automorphisms of } A \ltimes M \text{ of the form} \\ a \mapsto a + f(a)\varepsilon,\ m\varepsilon \mapsto m\varepsilon \end{array} \right\},$$

where we view $A \ltimes M \cong A + M\varepsilon$.

Moreover, the inner derivations correspond to automorphisms achieved by conjugation by $1 + m\varepsilon$, which is a unit with inverse $1 - m\varepsilon$.

**Proposition.** There is a bijection between $HH^2(A, M)$ with the isomorphism classes of extensions of $A$ by $M$.

*Proof.* Let $B$ be an extension with, as usual, $\pi : B \to A$, $I = M = \ker \pi$, $I^2 = 0$. We now try to produce a cocycle from this.

Let $\rho$ be any $k$-linear map $A \to B$ such that $\pi(\rho(a)) = a$. This is possible since $\pi$ is surjective. Equivalently, $\rho(\pi(b)) = b \bmod I$. We define a $k$-linear map

$$f_\rho : A \otimes A \to I \cong M$$

by

$$a_1 \otimes a_2 \mapsto \rho(a_1)\rho(a_2) - \rho(a_1 a_2).$$

Note that the image lies in $I$ since

$$\rho(a_1)\rho(a_2) \equiv \rho(a_1 a_2) \pmod{I}.$$

It is a routine check that $f_\rho$ is a 2-cocycle, i.e. it lies in $\ker \delta_2$.

If we replace $\rho$ by any other $\rho'$, we get $f_{\rho'}$, and we have

$$\begin{aligned}
&f_\rho(a_1 \otimes a_2) - f_{\rho'}(a_1 \otimes a_2)\\
&= \rho(a_1)(\rho(a_2) - \rho'(a_2)) - (\rho(a_1 a_2) - \rho'(a_1 a_2)) + (\rho(a_1) - \rho'(a_1))\rho'(a_2)\\
&= a_1 \cdot (\rho(a_2) - \rho'(a_2)) - (\rho(a_1 a_2) - \rho'(a_1 a_2)) + (\rho(a_1) - \rho'(a_1)) \cdot a_2,
\end{aligned}$$

where $\cdot$ denotes the $A$-$A$-bimodule action in $I$. Thus, we find

$$f_\rho - f_{\rho'} = \delta_1(\rho - \rho'),$$

noting that $\rho - \rho'$ actually maps to $I$.

So we obtain a map from the isomorphism classes of extensions to the second cohomology group.

Conversely, given an $A$-$A$-bimodule $M$ and a 2-cocycle $f : A \otimes A \to M$, we let

$$B_f = A \oplus M$$

as $k$-vector spaces. We define the multiplication map

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2 + f(a_1 \otimes a_2)).$$

This is associative precisely because of the 2-cocycle condition. The map $(a, m) \to a$ yields a homomorphism $\pi : B \to A$, with kernel $I$ being a two-sided ideal of $B$ which has $I^2 = 0$. Moreover, $I \cong M$ as an $A$-$A$-bimodule. Taking $\rho : A \to B$ by $\rho(a) = (a, 0)$ yields the 2-cocycle we started with.

Finally, let $f'$ be another 2 co-cycle cohomologous to $f$. Then there is a linear map $\tau : A \to M$ with

$$f - f' = \delta_1 \tau.$$

That is,

$$f(a_1 \otimes A_2) = f'(a_1 \otimes a_2) + a_1 \cdot \tau(a_2) - \tau(a_1 a_2) + \tau(a_1) \cdot a_2.$$

Then consider the map $B_f \to B_f'$ given by

$$(a, m) \mapsto (a, m + \tau(a)).$$

One then checks this is an isomorphism of extensions. And then we are done. $\square$

**Corollary.** If $HH^2(A, M) = 0$, then all extensions are split.

**Theorem** (Wedderburn, Malcev)**.** Let $B$ be a $k$-algebra satisfying

- $\mathrm{Dim}(B/J(B)) \leq 1$.
- $J(B)^2 = 0$

Then there is an subalgebra $A \cong B/J(B)$ of $B$ such that

$$B = A \ltimes J(B).$$

Furthermore, if $\mathrm{Dim}(B/J(B)) = 0$, then any two such subalgebras $A, A'$ are conjugate, i.e. there is some $x \in J(B)$ such that

$$A' = (1 + x)A(1 + x)^{-1}.$$

Notice that $1 + x$ is a unit in $B$.

*Proof.* We have $J(B)^2 = 0$. Since we know $\mathrm{Dim}(B/J(B)) \leq 1$, we must have

$$HH^2(A, J(B)) = 0$$

where

$$A \cong \frac{B}{J(B)}.$$

Note that we regard $J(B)$ as an $A$-$A$-bimodule here. So we know that all extension of $A$ by $J(B)$ are semi-direct, as required.

Furthermore, if $\mathrm{Dim}(B/J(B)) = 0$, then we know $HH^1(A, J(A)) = 0$. So by our older lemmas, we see that complements are all conjugate, as required. □

**Corollary.** If $k$ is algebraically closed and $\dim_k B < \infty$, then there is a subalgebra $A$ of $B$ such that

$$A \cong \frac{B}{J(B)},$$

and

$$B = A \ltimes J(B).$$

Moreover, $A$ is unique up to conjugation by units of the form $1+x$ with $x \in J(B)$.

*Proof.* We need to show that $\mathrm{Dim}(A) = 0$. But we know $B/J(B)$ is a semi-simple $k$-algebra of finite dimension, and in particular is Artinian. So by Artin–Wedderburn, we know $B/J(B)$ is a direct sum of matrix algebras over $k$ (since $k$ is algebraically closed and $\dim_k(B/J(B))$).

We have previously observed that $M_n(k)$ is $k$-separable. Since $k$-separability behaves well under direct sums, we know $B/J(B)$ is $k$-separable, hence has dimension zero.

It is a general fact that $J(B)$ is nilpotent. □

## 3.3 Star products

**Theorem** (Gerstenhaber)**.** If $HH^3(A, A) = 0$, then all infinitesimal deformations are integrable.

**Theorem** (Gerstenhaber)**.** Any non-trivial star product $f$ is equivalent to one of the form

$$g(a, b) = ab + t^n G_n(a, b) + t^{n+1} G_{n+1}(a, b) + \cdots ,$$

where $G_n$ is a 2-cocycle and not a coboundary. In particular, if $HH^2(A, A) = 0$, then any star product is trivial.

*Proof.* Suppose as usual

$$f(a, b) = ab + tF_1(a, b) + t^2 F_2(a, b) + \cdots ,$$

and suppose $F_1, \cdots , F_{n-1} = 0$. Then it follows from (†) that

$$\delta_2 F_n = 0.$$

If $F_n$ is a coboundary, then we can write

$$F_n = -\delta\phi_n$$

for some $\phi_n : A \to A$. We set

$$\Phi_n(a) = a + t^n \phi_n(a).$$

Then we can compute that

$$\Phi_n^{-1}(f(\Phi_n(a), \Phi_n(b)))$$

is of the form

$$ab + t^{n+1} G_{n+1}(a, b) + \cdots .$$

So we have managed to get rid of a further term, and we can keep going until we get the first non-zero term not a coboundary.

Suppose this never stops. Then $f$ is trivial — we are using that $\cdots \circ \Phi_{n+2} \circ \Phi_{n+1} \circ \Phi_n$ converges in the automorphism ring, since we are adding terms of higher and higher degree. □

**Theorem** (Gerstenhaber)**.** Suppose $HH^2(A, A) = 0$. Then all derivations are integrable.

## 3.4 Gerstenhaber algebra

**Lemma.** The cup product on $HH^\bullet(A, A)$ is graded commutative, i.e.

$$f \smile g = (-1)^{mn}(g \smile f).$$

when $f \in HH^m(A, A)$ and $g \in HH^n(A, A)$.

*Proof.* We previously "noticed" that

$$(-1)^m(g \smile f - (-1)^{mn}(f \smile g)) = \delta(f \circ g),$$ □

**Theorem** (Hochschild–Kostant–Ronsenberg (HKR) theorem)**.** If $A$ is a "smooth" commutative $k$-algebra, and $\operatorname{char} k = 0$, then the canonical map

$$\bigwedge_A (\mathrm{Der} A) \to HH^*(A, A)$$

is an isomorphism of Gerstenhaber algebras.

**Theorem** (Kontsevich)**.** There is a bijection

$$\left\{ \begin{array}{c} \text{equivalence classes} \\ \text{of star products} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{classes of formal} \\ \text{Poisson structures} \end{array} \right\}$$

This applies for smooth algebras in $\operatorname{char} 0$, and in particular for polynomial algebras $A = k[X_1, \cdots, X_n]$.

## 3.5 Hochschild homology

**Lemma.**

$$HH_0(A, M) = \frac{M}{\langle xm - mx : m \in M, x \in A\rangle}.$$

In particular,

$$HH_0(A, A) = \frac{A}{[A, A]}.$$

*Proof.* Exercise. □

# 4 Coalgebras, bialgebras and Hopf algebras

**Lemma.** If $C$ is a coalgebra, then $C^*$ is an algebra with multiplication $\Delta^*$ (that is, $\Delta^*|_{C^* \otimes C^*}$) and unit $\varepsilon^*$. If $C$ is co-commutative, then $C^*$ is commutative.

**Theorem** (Mastnak, Witherspoon (2008))**.** The bialgebra cohomology $H^\bullet_{bi}(H, H)$ for a finite-dimensional Hopf algebra is equal to $HH^\bullet(D(H), k)$, where $k$ is the trivial module, and $D(H)$ is the Drinfeld double.

**Theorem** (Gerstenhaber–Schack)**.** Every deformation is equivalent to one where the unit and counit are unchnaged. Also, deformation preserves the existence of an antipode, though it might change.

**Theorem** (Gerstenhaber–Schack)**.** All deformations of $\mathcal{O}(M_n(k))$ or $\mathcal{O}(\mathrm{SL}_n(k))$ are equivalent to one in which the comultiplication is unchanged.