Part III — Algebras
Theorems

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The aim of the course is to give an introduction to algebras. The emphasis will be on non-commutative examples that arise in representation theory (of groups and Lie algebras) and the theory of algebraic D-modules, though you will learn something about commutative algebras in passing.

Topics we discuss include:

- Artinian algebras. Examples, group algebras of finite groups, crossed products. Structure theory. Artin–Wedderburn theorem. Projective modules. Blocks. K_0.
- Noetherian algebras. Examples, quantum plane and quantum torus, differential operator algebras, enveloping algebras of finite dimensional Lie algebras. Structure theory. Injective hulls, uniform dimension and Goldie's theorem.
- Hochschild chain and cochain complexes. Hochschild homology and cohomology. Gerstenhaber algebras.
- Deformation of algebras.
- Coalgebras, bialgebras and Hopf algebras.

Pre-requisites

It will be assumed that you have attended a first course on ring theory, eg IB Groups, Rings and Modules. Experience of other algebraic courses such as II Representation Theory, Galois Theory or Number Fields, or III Lie algebras will be helpful but not necessary.

Contents

0 Introduction

Theorem (Artin–Wedderburn theorem)**.** Let A be a left-Artinian algebra such that the intersection of the maximal left ideals is zero. Then A is the direct sum of finitely many matrix algebras over division algebras.

Theorem (Goldie's theorem)**.** Let A be a right Noetherian algebra with no non-zero ideals all of whose elements are nilpotent. Then A embeds in a finite direct sum of matrix algebras over division algebras.

1 Artinian algebras

1.1 Artinian algebras

Proposition. Let A be an algebra and I a left ideal. Then I is a maximal left ideal iff A/I is simple.

Proposition. Let A be an algebra and M a simple module. Then $M \cong A/I$ for some (maximal) left ideal I of A.

Lemma. Let M be a finitely-generated A module. Then M has a maximal proper submodule M'.

Lemma (Nakayama lemma)**.** The following are equivalent for a left ideal I of A.

(i) $I \leq J(A)$.

(ii) For any finitely-generated left A-module M, if $IM = M$, then $M = 0$, where IM is the module generated by elements of the form am, with $a \in I$ and $m \in M$.

(iii) $G = \{1 + a : a \in I\} = 1 + I$ is a subgroup of the unit group of A.

Proposition. Let M be an A-module. Then the following are equivalent:

(i) M is completely reducible.

(ii) M is the direct sum of simple modules.

(iii) Every submodule of M has a *complement*, i.e. for any submodule N of M, there is a complement N' such that $M = N \oplus N'$.

Proposition. Sums, submodules and quotients of completely reducible modules are completely reducible.

Proposition. Let M be an A-module satisfying the descending chain condition on submodules. Then M is completely reducible iff $\mathrm{Rad}(M) = 0$.

Corollary. If A is a semi-simple left Artinian algebra, then ${}_AA$ is completely reducible.

Corollary. If A is a semi-simple left Artinian algebra, then every left A-module is completely reducible.

Lemma. Let A be left Artinian, and M a finitely generated left A-module, then $J(A)M = \mathrm{Rad}(M)$.

Proposition. Let A be left Artinian. Then

(i) $J(A)$ is nilpotent, i.e. there exists some r such that $J(A)^r = 0$.

(ii) If M is a finitely-generated left A-module, then it is both left Artinian and left Noetherian.

(iii) A is left Noetherian.

1.2 Artin–Wedderburn theorem

Theorem (Artin–Wedderburn theorem)**.** Let A be a semisimple right Artinian algebra. Then

$$A = \bigoplus_{i=1}^{r} M_{n_i}(D_i),$$

for some division algebra D_i, and these factors are uniquely determined.

A has exactly r isomorphism classes of simple (right) modules S_i, and

$$\mathrm{End}_A(S_i) = \{A\text{-module homomorphisms } S_i \to S_i\} \cong D_i,$$

and

$$\dim_{D_i}(S_i) = n_i.$$

If A is simple, then $r = 1$.

Lemma (Schur's lemma)**.** Let M_1, M_2 be simple right A-modules. Then either $M_1 \cong M_2$, or $\mathrm{Hom}_A(M_1, M_2) = 0$. If M is a simple A-module, then $\mathrm{End}_A(M)$ is a division algebra.

Lemma.

(i) If M is a right A-module and e is an idempotent in A, i.e. $e^2 = e$, then $Me \cong \mathrm{Hom}_A(eA, M)$.

(ii) We have
$$eAe \cong \mathrm{End}_A(eA).$$
In particular, we can take $e = 1$, and recover $\mathrm{End}_A(A_A) \cong A$.

Lemma. Let M be a completely reducible right A-module. We write

$$M = \bigoplus S_i^{n_i},$$

where $\{S_i\}$ are distinct simple A-modules. Write $D_i = \mathrm{End}_A(S_i)$, which we already know is a division algebra. Then

$$\mathrm{End}_A(S_i^{n_i}) \cong M_{n_i}(D_i),$$

and

$$\mathrm{End}_A(M) = \bigoplus M_{n_i}(D_i)$$

Corollary. If k is algebraically closed and A is a finite-dimensional semi-simple k-algebra, then

$$A \cong \bigoplus M_{n_i}(k).$$

Theorem (Maschke's theorem)**.** Let G be a finite group and $p \nmid |G|$, where $p = \mathrm{char}\, k$, so that $|G|$ is invertible in k, then kG is semi-simple.

Theorem. Let G be finite and kG semi-simple. Then $\mathrm{char}\, k \nmid |G|$.

Theorem. Let k be algebraically closed of characteristic p, and G be finite. Then the number of simple kG modules (up to isomorphism) is equal to the number of conjugacy classes of elements of order not divisible by p. These are known as the *p-regular elements*.

Corollary. If $|G| = p^r$ for some r and p is prime, then the trivial module is the only simple kG module, when $\mathrm{char}\, k = p$.

1.3 Crossed products

1.4 Projectives and blocks

Lemma. The following are equivalent:

(i) P is projective.

(ii) Every surjective map $\phi : M \to P$ splits, i.e.

$$M \cong \ker \phi \oplus N$$

where $N \cong P$.

(iii) P is a direct summand of a free module.

Theorem (Krull–Schmidt theorem)**.** Suppose M is a finite sum of indecomposable A-modules M_i, with each $\text{End}(M_i)$ local. Then M has the unique decomposition property.

Lemma (Fitting)**.** Suppose M is a module with both the ACC and DCC on submodules, and let $f \in \text{End}_A(M)$. Then for large enough n, we have

$$M = \text{im } f^n \oplus \ker f^n.$$

Lemma. Suppose M is an indecomposable module satisfying ACC and DCC on submodules. Then $B = \text{End}_A(M)$ is local.

Corollary. Let A be a left Artinian algebra. Then A has the unique decomposition property.

Proposition. Let N be a nilpotent ideal in A, and let f be an idempotent of $A/N \equiv \bar{A}$. Then there is an idempotent $e \in A$ with $f = \bar{e}$.

Corollary. Let N be a nilpotent ideal of A. Let

$$\bar{1} = f_1 + \cdots + f_r$$

with $\{f_i\}$ orthogonal primitive idempotents in A/N. Then we can write

$$1 = e_1 + \cdots + e_r,$$

with $\{e_i\}$ orthogonal primitive idempotents in A, and $\bar{e}_i = f_i$.

Lemma. Let P be an indecomposable projective, and M an A-module. Then $\text{Hom}(P, M) \neq 0$ iff $P/PJ(A)$ is a composition factor of M.

Theorem. Indecomposable projectives P_1 and P_2 are in the same block if and only if they lie in the same connected component of the graph.

1.5 K_0

2 Noetherian algebras

2.1 Noetherian algebras

Theorem (Hilbert basis theorem)**.** If A is Noetherian, then $A[X]$ is Noetherian.

Theorem. Let A be left Noetherian. Then $A[[X]]$ is Noetherian.

Lemma. Let A be a positively filtered algebra. If $\operatorname{gr} A$ is Noetherian, then A is left Noetherian.

Corollary. $A_n(k)$ and $\mathcal{U}(\mathfrak{g})$ are left/right Noetherian.

2.2 More on $A_n(k)$ and $\mathcal{U}(\mathfrak{g})$

Lemma. Suppose $\operatorname{char} k = 0$. Then $A_n(k)$ has no non-zero modules that are finite-dimensional k-vector spaces.

Theorem (Hilbert-Serre theorem)**.** The Poincaré series $P(V, t)$ of a finitely-generated graded module

$$V = \bigoplus_{i=0}^{\infty} V_i$$

over a finitely-generated generated commutative algebra

$$S = \bigoplus_{i=0}^{\infty} S_i$$

with homogeneous generating set $x_1, \cdots, x_m$ is a rational function of the form

$$\frac{f(t)}{\prod(1 - t^{k_i})},$$

where $f(t) \in \mathbb{Z}[t]$ and k_i is the degree of the generator x_i.

Corollary. If each $k_1, \cdots, k_m = 1$, i.e. S is generated by $S_0 = k$ and homogeneous elements $x_1, \cdots, x_m$ of degree 1, then for large enough i, then $\dim V_i = \phi(i)$ for some polynomial $\phi(t) \in \mathbb{Q}[t]$ of $d - 1$, where d is the order of the pole of $P(V, t)$ at $t = 1$. Moreover,

$$\sum_{j=0}^{i} \dim V_j = \chi(i),$$

where $\chi(t) \in \mathbb{Q}[t]$ of degree d.

Lemma. Let M be a finitely-generated A_n-module. Then $d(M) \leq 2n$.

Theorem (Bernstein's inequality)**.** Let M be a non-zero finitely-generated $A_n(k)$-module, and $\operatorname{char} k = 0$. Then

$$d(M) \geq n.$$

2.3 Injective modules and Goldie's theorem

Theorem (Goldie's theorem)**.** Let A be a right Noetherian algebra with no non-zero ideals all of whose elements are nilpotent. Then A embeds in a finite direct sum of matrix algebras over division algebras.

Lemma. Every direct summand of an injective module is injective, and direct products of injectives is injective.

Lemma. Every A-module may be embedded in an injective module.

Lemma. An A-module is injective iff it is a direct summand of every extension of itself.

Lemma. An essential extension of an essential extension is essential.

Lemma. A maximal essential extension is an injective module.

Proposition. Let M be an A-module, with an inclusion $M \hookrightarrow I$ into an injective module. Then this extends to an inclusion $E(M) \hookrightarrow I$.

Proposition. Suppose E is an injective essential extension of M. Then $E \cong E(M)$. In particular, any two injective hulls are isomorphic.

Proposition.
$$E(M_1 \oplus M_2) = E(M_1) \oplus E(M_2).$$

Lemma. V is uniform iff $E(V)$ is indecomposable.

Lemma. Let A be a filtered algebra, which is exhaustive and separated. Then if $\operatorname{gr} A$ is a domain, then so is A.

Corollary. $A_n(k)$ and $\mathcal{U}(\mathfrak{g})$ are domains.

Lemma. Let A be a right Noetherian domain. Then A_A is uniform, i.e. $E(A_A)$ is indecomposable.

Lemma. Let E be an indecomposable injective right module. Then $\operatorname{End}_A(E)$ is a local algebra, with the unique maximal ideal given by
$$I = \{f \in \operatorname{End}(E) : \ker f \text{ is essential}\}.$$

Lemma. Let M be a non-zero Noetherian module. Then M is an essential extension of a direct sum of uniform submodules $N_1, \cdots, N_r$. Thus
$$E(M) \cong E(N_1) \oplus \cdots E(N_r)$$
is a direct sum of finitely many indecomposables.

This decomposition is unique up to re-ordering (and isomorphism).

Lemma. Let $E_1, \cdots, E_r$ be indecomposable injectives. Put $E = E_1 \oplus \cdots \oplus E_r$. Let $I = \{f \in \operatorname{End}_A(E) : \ker f \text{ is essential}\}$. This is an ideal, and then
$$\operatorname{End}_A(E)/I \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_s}(D_s)$$
for some division algebras D_i.

Lemma. If A is a right Noetherian algebra, then any $f : A_A \to A_A$ with $\ker f$ essential in A_A is nilpotent.

Theorem (Goldie's theorem)**.** Let A be a right Noetherian algebra with no non-zero ideals all of whose elements are nilpotent. Then A embeds in a finite direct sum of matrix algebras over division algebras.

3 Hochschild homology and cohomology

3.1 Introduction

3.2 Cohomology

Lemma. Let M be an injective bimodule. Then $HH^n(A, M) = 0$ for all $n \geq 1$.

Lemma. If ${}_AA_A$ is a projective bimodule, then $HH^n(A, M) = 0$ for all M and all $n \geq 1$.

Lemma. If $\operatorname{Dim} A = 0$, then A is separable.

Proposition. We have

$$HH^0(A, M) = \{m \in M : am - ma = 0 \text{ for all } a \in A\}.$$

In particular, $HH^0(A, A)$ is the center of A.

Proposition.

$$\ker \delta_1 = \{f \in \operatorname{Hom}_k(A, M) : f(a_1 a_2) = a_1 f(a_2) + f(a_1) a_2\}.$$

These are the *derivations* from A to M. We write this as $\operatorname{Der}(A, M)$.

On the other hand,

$$\operatorname{im} \delta_0 = \{f \in \operatorname{Hom}_k(A, M) : f(a) = am - ma \text{ for some } m \in M\}.$$

These are called the *inner derivations* from A to M. So

$$HH^1(A, M) = \frac{\operatorname{Der}(A, M)}{\operatorname{InnDer}(A, M)}.$$

Setting $A = M$, we get the derivations and inner derivations of A.

Lemma. We have

$$\operatorname{Der}_k(A, M) \cong \{\text{algebra complements to } M \text{ in } A \ltimes M \text{ isomorphic to } A\}.$$

Lemma. We have

$$\operatorname{Der}(A, M) \cong \left\{\begin{array}{c}\text{automorphisms of } A \ltimes M \text{ of the form}\\ a \mapsto a + f(a)\varepsilon,\ m\varepsilon \mapsto m\varepsilon\end{array}\right\},$$

where we view $A \ltimes M \cong A + M\varepsilon$.

Moreover, the inner derivations correspond to automorphisms achieved by conjugation by $1 + m\varepsilon$, which is a unit with inverse $1 - m\varepsilon$.

Proposition. There is a bijection between $HH^2(A, M)$ with the isomorphism classes of extensions of A by M.

Corollary. If $HH^2(A, M) = 0$, then all extensions are split.

Theorem (Wedderburn, Malcev)**.** Let B be a k-algebra satisfying

- $\operatorname{Dim}(B/J(B)) \leq 1$.
- $J(B)^2 = 0$

Then there is an subalgebra $A \cong B/J(B)$ of B such that

$$B = A \ltimes J(B).$$

Furthermore, if $\mathrm{Dim}(B/J(B)) = 0$, then any two such subalgebras A, A' are conjugate, i.e. there is some $x \in J(B)$ such that

$$A' = (1 + x)A(1 + x)^{-1}.$$

Notice that $1 + x$ is a unit in B.

Corollary. If k is algebraically closed and $\dim_k B < \infty$, then there is a subalgebra A of B such that

$$A \cong \frac{B}{J(B)},$$

and

$$B = A \ltimes J(B).$$

Moreover, A is unique up to conjugation by units of the form $1+x$ with $x \in J(B)$.

3.3 Star products

Theorem (Gerstenhaber)**.** If $HH^3(A, A) = 0$, then all infinitesimal deformations are integrable.

Theorem (Gerstenhaber)**.** Any non-trivial star product f is equivalent to one of the form

$$g(a, b) = ab + t^n G_n(a, b) + t^{n+1} G_{n+1}(a, b) + \cdots ,$$

where G_n is a 2-cocycle and not a coboundary. In particular, if $HH^2(A, A) = 0$, then any star product is trivial.

Theorem (Gerstenhaber)**.** Suppose $HH^2(A, A) = 0$. Then all derivations are integrable.

3.4 Gerstenhaber algebra

Lemma. The cup product on $HH^\bullet(A, A)$ is graded commutative, i.e.

$$f \smile g = (-1)^{mn}(g \smile f).$$

when $f \in HH^m(A, A)$ and $g \in HH^n(A, A)$.

Theorem (Hochschild–Kostant–Ronsenberg (HKR) theorem)**.** If A is a "smooth" commutative k-algebra, and $\mathrm{char}\, k = 0$, then the canonical map

$$\bigwedge_A (\mathrm{Der} A) \to HH^*(A, A)$$

is an isomorphism of Gerstenhaber algebras.

Theorem (Kontsevich)**.** There is a bijection

$$\left\{ \begin{array}{c} \text{equivalence classes} \\ \text{of star products} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{classes of formal} \\ \text{Poisson structures} \end{array} \right\}$$

This applies for smooth algebras in $\mathrm{char}\, 0$, and in particular for polynomial algebras $A = k[X_1, \cdots, X_n]$.

3.5 Hochschild homology

Lemma.

$$HH_0(A, M) = \frac{M}{\langle xm - mx : m \in M, x \in A\rangle}.$$

In particular,

$$HH_0(A, A) = \frac{A}{[A, A]}.$$

4 Coalgebras, bialgebras and Hopf algebras

Lemma. If C is a coalgebra, then C^* is an algebra with multiplication Δ^* (that is, $\Delta^*|_{C^* \otimes C^*}$) and unit ε^*. If C is co-commutative, then C^* is commutative.

Theorem (Mastnak, Witherspoon (2008))**.** The bialgebra cohomology $H^{\bullet}_{bi}(H, H)$ for a finite-dimensional Hopf algebra is equal to $HH^{\bullet}(D(H), k)$, where k is the trivial module, and $D(H)$ is the Drinfeld double.

Theorem (Gerstenhaber–Schack)**.** Every deformation is equivalent to one where the unit and counit are unchnaged. Also, deformation preserves the existence of an antipode, though it might change.

Theorem (Gerstenhaber–Schack)**.** All deformations of $\mathcal{O}(M_n(k))$ or $\mathcal{O}(\mathrm{SL}_n(k))$ are equivalent to one in which the comultiplication is unchanged.