III Algebras Lent Term 2017

EXAMPLE SHEET 1

k is a field; A is a k-algebra.

1. Let G be a group. Show that $(kG)^{\mathrm{op}} \cong kG$, and that $(M_n(k))^{\mathrm{op}} \cong M_n(k)$.

2. Let $A = kG$. Let $g_1, \dots, g_m$ be conjugacy class representatives of G. Show that the $g_i + [A, A]$ form a k-vector space basis of $A/[A, A]$.

3. Let $\operatorname{char} k = p$. Show that when $A = kG$, $[A, A] + J(A)$ consists of those elements $x \in A$ such that $x^{p^i} \in [A, A]$ for some i.

4. Let k be algebraically closed of characteristic 0. Show that $\dim_k Z(kG)$ is the number of conjugacy classes in G, and is the number of isomorphism classes of simple kG-modules.

5. Show that the following are equivalent for an A-module P:

   (i) P is projective;

   (ii) Every surjective map $\phi : M \to P$ splits; and

   (iii) P is a direct summand of a free module.

6. Let A be an Artinian algebra. Let P_1 and P_2 be indecomposable projectives with

$$\frac{P_1}{P_1 J(A)} \cong \frac{P_2}{P_2 J(A)}.$$

   Show that $P_1 \cong P_2$.

7. A finite dimensional algebra A is *Frobenius* if there is a k-linear map $\lambda : A \to k$ such that $\ker \lambda$ contains no non-zero left or right ideal. Furthermore it is *symmetric* if $\lambda(xy) = \lambda(yx)$ for all $x, y \in A$.

   (i) Show that $M_n(k)$ is a Frobenius algebra, where λ is the trace map.

   (ii) Show that kG, for G finite, is Frobenius, where

$$\lambda\left(\sum a_g g\right) = a_e,$$

   where e is the identity element of G.

   (iii) Show that if A is a Frobenius algebra then the vector space dual $({}_A A)^* \cong A_A$. (Note that if M is a left module, then the dual has a natural structure as a right module via

$$(fa)(m) = f(am)$$

   where $f : M \to k \in M^*$.)

   (iv) Suppose P is an indecomposable projective for a symmetric Frobenius algebra A. Show that P has a unique simple submodule S with $S \cong P/PJ(A)$.

8. Let $k \leq K$ be a finite field extension which is Galois. Let $G = \mathrm{Gal}(K/k)$.

   (i) Establish the condition on

$$\Psi : G \times G \to K^\times$$

   that ensures the associativity of the crossed product (K, G, Ψ).

(ii) Suppose there is an isomorphism

$$\Theta : (K, G, \Psi_1) \to (K, G, \Psi_2)$$

which preserves the G-grading, and $\Theta|_K$ is the identity map on K. What can be said about the map $\Psi_3 : G \times G \to K$, where

$$\Psi_3(g, h) = \Psi_1(g, h)(\Psi_2(g, h))^{-1}\,?$$

9. Let k be algebraically closed. Let $G = S_3$. Decompose kG into blocks and into indecomposable projectives, where

   (i) $\operatorname{char} k = 2$; and

   (ii) $\operatorname{char} k = 3$.

10. Let $G = \mathrm{SL}_2(p)$. Show that G has exactly p conjugacy classes of elements of order not divisible by p. Let $\operatorname{char} k = p$.

    Let V_1 be the trivial kG-module. For $n \geq 2$, let V_n be the space of homogeneous polynomials in $k[X, Y]$ of total degree $n - 1$.

    Show that $\dim V_n = n$.

    Let kG act on V_2 by the canonical action of G on a 2-dimensional k-vector space. This induces a kG-module structure on V_n. Show that $V_1, \ldots, V_p$ are simple non-isomorphic kG-modules.

III Algebras Lent Term 2017

EXAMPLE SHEET 2

1. Let B be an algebra containing a subalgebra A. Suppose A is left Noetherian and let $x \in B$.
   (i) Suppose $A + xA = A + Ax$, and B is generated by A and x. Show that B is left Noetherian.
   (ii) Suppose there exists an automorphism σ of A such that $ax = xa^\sigma$ for all $a \in A$. Show that if B is generated by A and x, then it is left Noetherian.
   (iii) Suppose x is a unit of B and that $x^{-1}Ax = A$. Suppose B is generated by A, x, x^{-1}. Show that B is left Noetherian.

2. Show that
$$\begin{pmatrix} k[x] & k(x) \\ 0 & k(x) \end{pmatrix}$$
is right, but not left Noetherian.

3. An ideal P is *prime* if $I_1, I_2 \leq P$ implies $I_1 \leq P$ or $I_2 \leq P$ for ideals I_1 and I_2. Let A be a Noetherian algebra. Show that any ideal I contains a product of finitely many prime ideals. Let N be the intersection of all the prime ideals of A. Show that N is nilpotent. A *nil ideal* is one all of whose elements are nilpotent. Show that $I \leq N$.

4. Show that the quantum torus has no non-zero modules that are finite-dimensional as k-vector spaces.

5. Show carefully that $\operatorname{gr} A$ is commutative when $A = A_n(k)$ and $A = \mathcal{U}(\mathcal{L})$ with the usual filtrations. Show that $\operatorname{gr} A$ is a polynomial algebra in both cases.

6. Let I be an ideal of a commutative Noetherian algebra A. Show how the powers of I yield a negative filtration of A and that $\operatorname{gr} A$ is Noetherian. Is the converse true, that $\operatorname{gr} A$ Noetherian $\Rightarrow$ A is Noetherian?

7. Show that $A_n(k)$ is a simple algebra, i.e. its only ideals are 0 and $A_n(k)$.

8. Show that GK-dimension is independent of the choices of generating sets.

9. Let f be a homogeneous non-zero element (of degree t) of a graded commutative algebra S generated by degree 1 elements. What is the Samuel polynomial of $S/(f)$? Show that
$$d(S/(f)) = d(S) - 1.$$

10. What is the GK-dimension of the quantum torus $k_q[X, X^{-1}, Y, Y^{-1}]$?

11. Find a holonomic $A_n(k)$-module M other than $k[X_1, \ldots, X_n]$. (ie. $d(M) = n$)

12. Let $A = k[x]$ and k be the trivial A-module, where X acts like 0. Show that the injective hull
$$E(k) \cong \frac{k[X, X^{-1}]}{Xk[X]}.$$

13. Show that the injective hull of a module is unique up to isomorphism.

III Algebras Lent Term 2017

EXAMPLE SHEET 3

1. Show that if A is a separable k-algebra, then $\dim_k(A) < \infty$.
2. Show that $\sum_{i,j} E_{ij} \otimes E_{ji}$ is a separating idempotent for $M_n(k)$, where E_{ij} is the elementary matrix with 1 in the ijth entry and 0 otherwise.
3. Show that every separable k-algebra is semisimple.
4. Show that the derivations $A \to A$ form a Lie algebra and that the inner derivations form a Lie ideal.
5. What are the derivations of $k[X]$ when $\operatorname{char} k = p \neq 0$?
6. Define the differential operators on an algebra A inductively:

$$\begin{aligned} D^0(A) &= \{D \in \operatorname{End}_k(A) : [x, D] = 0 \ \forall x \in A\} \\ D^1(A) &= \{D \in \operatorname{End}_k(A) : [x, D] \in D^0(A)\} \\ D^2(A) &= \{D \in \operatorname{End}_k(A) : [x, D] \in D^1(A)\} \end{aligned}$$

   etc. Set

$$\operatorname{Diff}(A) = \bigcup_{i=0}^{\infty} D^i(A).$$

   Let $\operatorname{char} k = 0$. Show that

$$A_n(k) = \operatorname{Diff}(k[X_1, \ldots, X_n]).$$

7. Show that if $HH^2(A, A) = 0$, then all derivations $\phi_1 : A \to A$ are integrable.
8. Calculate $HH^*(A, A)$ for $A = k[X, Y]$ when $\operatorname{char} k = 0$.
9. Calculate $HH^*(A, A)$ when $A = kG$, where k is algebraically closed of characteristic 2 and G cyclic group of order 2.
10. Calculate $HH^*(A, A)$ where $A = kS_3$ and k algebraically closed of characteristic 2.
11. For a co-algebra C, define a *right C co-module* to be a k-vector space V with a k-linear map $\rho : V \to V \otimes C$ such that

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes C \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ V \otimes C & \xrightarrow{\rho\otimes\mathrm{id}} & V \otimes C \otimes C \end{array} \quad , \quad \begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes C \\ & \searrow{\scriptstyle\sim} & \downarrow{\scriptstyle \mathrm{id}\otimes\varepsilon} \\ & & V \otimes k \end{array}$$

   commute. Show that if (V, ρ) is a right C co-module, then V is a left C^*-module.
12. Show that if B is a bialgebra and I is both an ideal and a co-ideal, then B/I is a bialgebra.
13. What are the prime ideals of $k_q[X, Y]$ (quantum plane) when k is algebraically closed and q is not a root of unity.
14. What are the prime ideals of $\mathcal{O}_q(\mathrm{SL}_2(k))$ when k is algebraically and q is not a root of unity.
15. Calculate $HH_*(A, A)$ for $A = k[X]$ and for $A = k[X, Y]$ when $\operatorname{char} k = 0$.