Part III — Advanced Quantum Field Theory

Theorems with proof

Based on lectures by D. B. Skinner

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Lent 2017

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course builds on the Michaelmas Quantum Field Theory course, using techniques of path integrals and functional methods to study quantum gauge theories. Gauge Theories are a generalisation of electrodynamics and form the backbone of the Standard Model — our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie Group at every point in space-time and yet still describe the same physics. Quantising a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry. We discuss the construction of gauge theories and their most important observables, Wilson Loops. Time permitting, we will explore the possibility that a classical symmetry may be broken by quantum effects. Such anomalies have many important consequences, from constraints on interactions between matter and gauge fields, to the ability to actually render a QFT inconsistent.

A further major component of the course is to study Renormalization. Wilson's picture of Renormalisation is one of the deepest insights into QFT — it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the Renormalisation Group (RG) flow. The course explores renormalisation systematically, from the use of dimensional regularisation in perturbative loop integrals, to the difficulties inherent in trying to construct a quantum field theory of gravity. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a non-Abelian gauge theory can effectively become small at high energies. Known as "asymptotic freedom", this phenomenon revolutionised our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrise possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Pre-requisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be very helpful.

Contents

0 Introduction

0.1 What is quantum field theory

0.2 Building a quantum field theory

1 QFT in zero dimensions

1.1 Free theories

Theorem (Wick's theorem)**.** For a monomial

$$P(\phi) = \prod_{i=1}^{2k} \ell_i(\phi),$$

we have

$$\langle P(\phi) \rangle = \hbar^k \sum_{\sigma \in \Pi_{2k}} \prod_{i \in \{1, \cdots, 2k\}/\sigma} M^{-1}(\ell_i, \ell_{\sigma(i)}).$$

where the $\{1, \cdots, 2k\}/\sigma$ says we sum over each pair $\{i, \sigma(i)\}$ only once, rather than once for $(i, \sigma(i))$ and another for $(\sigma(i), i)$.

1.2 Interacting theories

1.3 Feynman diagrams

1.4 An effective theory

1.5 Fermions

Proposition. For an invertible $n \times n$ matrix B and $\eta_i, \bar{\eta}_i, \theta^i, \bar{\theta}^i$ independent fermionic variables for $i = 1, \ldots, n$, we have

$$\mathcal{Z}(\eta, \bar{\eta}) = \int \mathrm{d}^n\theta \; \mathrm{d}^n\bar{\theta} \; \exp\left(\bar{\theta}^i B_{ij} \theta^j + \bar{\eta}_i \theta^i + \bar{\theta}^i \eta_i\right) = \det B \exp\left(\bar{\eta}_i (B^{-1})^{ij} \eta_j\right).$$

In particular, we have

$$\mathcal{Z} = \mathcal{Z}(0, 0) = \det B.$$

2 QFT in one dimension (i.e. QM)

2.1 Quantum mechanics

Theorem. There are no Lebesgue measures on an infinite dimensional inner product space.

2.2 Feynman rules

2.3 Effective quantum field theory

2.4 Quantum gravity in one dimension

3 Symmetries of the path integral

3.1 Ward identities

3.2 The Ward–Takahashi identity

4 Wilsonian renormalization

4.1 Background setting

Proposition.

$$[\partial_\mu] = 1, \quad [\varphi] = \frac{d-2}{2}.$$

4.2 Integrating out modes

4.3 Correlation functions and anomalous dimensions

4.4 Renormalization group flow

4.5 Taking the continuum limit

4.6 Calculating RG evolution

5 Perturbative renormalization

5.1 Cutoff regularization

5.2 Dimensional regularization

5.3 Renormalization of the ϕ^4 coupling

5.4 Renormalization of QED

6 Non-abelian gauge theory

6.1 Bundles, connections and curvature

Theorem. Given a principal G-bundle $\pi : P \to M$ and a representation $\rho : G \to \mathrm{GL}(V)$, there is a canonical way of producing a G-bundle $E \to M$ with fiber V. This is called the *associated bundle*.

Conversely, given a G-bundle $E \to M$ with fiber V, there is a canonical way of producing a principal G-bundle out of it, and these procedures are mutual inverses.

Moreover, this gives a correspondence between local trivializations of $P \to M$ and local trivializations of $E \to M$.

Proof. If the expression

$$P \times_G V \to M$$

makes any sense to you, then this proves the first part directly. Otherwise, just note that both a principal G-bundle and a G-bundle with fiber V can be specified just by the transition functions, which do not make any reference to what the fibers look like.

The proof is actually slightly less trivial than this, because the same vector bundle can have be many choices of trivializing covers, which gives us different transition functions. While these different transition functions patch to give the same vector bundle, by assumption, it is not immediate that they must give the same principal G-bundle as well, or vice versa.

The way to fix this problem is to figure out explicitly when two collection of transition functions give the same vector bundle or principal bundle, and the answer is that this is true if they are *cohomologous*. Thus, the precise statement needed to prove this is that both principal G-bundle and G-bundles with fiber V biject naturally with the *first Čech cohomology group* of M with coefficients in G. □

Theorem. There exists a notion of a connection on a principal G-bundle. Locally on a trivializing neighbourhood U, the connection 1-form is an element $A_\mu(x) \in \Omega^1_U(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

Every connection on a principal G-bundle induces a connection on any associated vector bundle. On local trivializations, the connection on the associated vector bundle has the "same" connection 1-form $A_\mu(x)$, where $A_\mu(x)$ is regarded as an element of $\mathrm{End}(V)$ by the action of G on the vector space.

6.2 Yang–Mills theory

6.3 Quantum Yang–Mills theory

6.4 Faddeev–Popov ghosts

Theorem. The integral $(*)$ is independent of the choice of f and C.

Proof. We first note that if we replace $f(\mathbf{x})$ by $c(r)f(\mathbf{x})$ for some $c > 0$, then we have

$$\delta(cf) = \frac{1}{|c|}\delta(f), \quad |\Delta_{cf}(x)| = c(r)|\Delta_f|,$$

and so the integral doesn't change.

Next, suppose we replace f with some $\tilde{f}$, but they have the same zero set. Now notice that $\delta(f)$ and $|\Delta_f|$ depend only on the first-order behaviour of f at C. In particular, it depends only on $\frac{\partial f}{\partial \theta}$ on C. So for all practical purposes, changing f to $\tilde{f}$ is equivalent to multiplying f by the ratio of their derivatives. So changing the function f while keeping C fixed doesn't affect the value of $(*)$.

Finally, suppose we have two arbitrary f and $\tilde{f}$, with potentially different zero sets. Now for each value of r, we pick a rotation $R_{\theta(r)} \in \mathrm{SO}(2)$ such that

$$\tilde{f}(x) \propto f(R_{\theta(r)}\mathbf{x}).$$

By the previous part, we can rescale f or $\tilde{f}$, and assume we in fact have equality.

We let $\mathbf{x}' = R_{\theta(r)}\mathbf{x}$. Now since the action only depends on the radius, it in particular is invariant under the action of $R_{\theta(r)}$. The measure $\mathrm{d}x\ \mathrm{d}y$ is also invariant, which is particularly clear if we write it as $\mathrm{d}\theta\ r\ \mathrm{d}r$ instead. Then we have

$$\begin{aligned}\int_{\mathbb{R}^2} \mathrm{d}x\ \mathrm{d}y\ \delta(f(\mathbf{x}))|\Delta_f(\mathbf{x})|e^{-S(\mathbf{x})} &= \int_{\mathbb{R}^2} \mathrm{d}x'\ \mathrm{d}y'\ \delta(f(\mathbf{x}'))|\Delta_f(\mathbf{x}')|e^{-S(\mathbf{x}')}\\ &= \int_{\mathbb{R}^2} \mathrm{d}x'\ \mathrm{d}y'\ \delta(\tilde{f}(\mathbf{x}))|\Delta_{\tilde{f}}(\mathbf{x})|e^{-S(\mathbf{x}')}\\ &= \int_{\mathbb{R}^2} \mathrm{d}x\ \mathrm{d}y\ \delta(\tilde{f}(\mathbf{x}))|\Delta_{\tilde{f}}(\mathbf{x})|e^{-S(\mathbf{x})}\end{aligned}$$ □

6.5 BRST symmetry and cohomology

Theorem. We have $\mathcal{Q}^2 = 0$.

Proof. We first check that for any field Ψ, we have $\mathcal{Q}^2\Psi = 0$.

- This is trivial for h.
- We have
$$\mathcal{Q}^2\bar{c} = \mathcal{Q}ih = 0.$$
- Note that for fermionic a, b, we have $[a, b] = [b, a]$. So
$$\mathcal{Q}^2 c = -\frac{1}{2}\mathcal{Q}[c, c] = -\frac{1}{2}([\mathcal{Q}c, c] + [c, \mathcal{Q}c]) = -[\mathcal{Q}c, c] = \frac{1}{2}[[c, c], c].$$
  It is an exercise to carefully go through the anti-commutativity and see that the Jacobi identity implies this vanishes.
- Noting that G acts on the ghosts fields by the adjoint action, we have
$$\begin{aligned}\mathcal{Q}^2 A_\mu &= \mathcal{Q}\nabla_\mu c\\ &= \nabla_\mu(\mathcal{Q}c) + [\mathcal{Q}A_\mu, c]\\ &= -\frac{1}{2}\nabla_\mu[c, c] + [\nabla_\mu c, c]\\ &= -\frac{1}{2}([\nabla_\mu c, c] + [c, \nabla_\mu c]) + [\nabla_\mu c, c]\\ &= 0.\end{aligned}$$

To conclude the proof, it suffices to show that if $a, b \in B$ are elements of definite statistics such that $\mathcal{Q}^2 a = \mathcal{Q}^2 b = 0$, then $\mathcal{Q}^2 ab = 0$. Then we are done by induction and linearity. Using the fact that $|\mathcal{Q}a| = |a| + 1 \pmod 2$, we have

$$\mathcal{Q}^2(ab) = (\mathcal{Q}^2 a)b + a\mathcal{Q}^2 b + (-1)^{|a|}(\mathcal{Q}a)(\mathcal{Q}b) + (-1)^{|a|+1}(\mathcal{Q}a)(\mathcal{Q}b) = 0. \quad \square$$

Theorem. The Yang–Mills Lagrangian is BRST closed. In other words, $\mathcal{Q}\mathcal{L} = 0$.

Proof. We first look at the $(F_{\mu\nu}, F^{\mu\nu})$ piece of $\mathcal{L}$. We notice that for the purposes of this term, since δA is bosonic, the BRST transformation for A looks just like a gauge transformation $A_\mu \mapsto A_\mu + \nabla_\mu \lambda$. So the usual (explicit) proof that this is invariant under gauge transformations shows that this is also invariant under BRST transformations.

We now look at the remaining terms. We claim that it is not just BRST closed, but in fact BRST exact. Indeed, it is just

$$\mathcal{Q}(\bar{c}^a f^a[A]) = ih^a f^a[A] - \bar{c}^a \frac{\delta f}{\delta \lambda} c. \quad \square$$

Lemma. Suppose δ is some operator such that $\Phi \mapsto \Phi + \delta\Phi$ is a symmetry. Then for all $\mathcal{O}$, we have

$$\langle \delta \mathcal{O} \rangle = 0.$$

Proof.

$$\langle \mathcal{O}(\Phi) \rangle = \langle \mathcal{O}(\Phi') \rangle = \langle \mathcal{O}(\Phi) + \delta\mathcal{O} \rangle. \quad \square$$

Corollary. Adding BRST exact terms to the Lagrangian does not affect the expectation of BRST invariant functions.

Proof. Suppose we add $\mathcal{Q}g$ to the Lagrangian, and $\mathcal{O}$ is BRST invariant. Then the change in $\langle \mathcal{O} \rangle$ is

$$\left\langle \int \mathcal{Q}g(x)\, \mathrm{d}^d x\, \mathcal{O} \right\rangle = \int \langle \mathcal{Q}(g\mathcal{O}) \rangle\, \mathrm{d}^d x = 0.$$

If we want to argue about this more carefully, we should use $\varepsilon\mathcal{Q}$ instead of $\mathcal{Q}$. $\square$

6.6 Feynman rules for Yang–Mills

6.7 Renormalization of Yang–Mills theory