Part III Maths **Lent Term 2017**
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Advanced Quantum Field Theory
Example Sheet 1

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (∗) may be more difficult.

1. Integration of a fermionic (*i.e.* anticommuting) variable θ is defined by the rules

$$\int \mathrm{d}\theta\, \theta = 1 \qquad \text{and} \qquad \int \mathrm{d}\theta\ 1 = 0\,,$$

often called *Berezinian integration.* If B is an invertible $n \times n$ matrix and θ^i, $\bar{\theta}^i$, η_i and $\bar{\eta}_i$ are independent fermionic variables ($i = 1, \ldots, n$), show that

$$Z(\eta, \bar{\eta}) := \int \mathrm{d}^n\theta\, \mathrm{d}^n\bar{\theta}\ \exp\left(\bar{\theta}^i B_{ij}\theta^j + \bar{\eta}_i\theta^i + \bar{\theta}^i\eta_i\right) = \det B\ \exp(\bar{\eta}_i (B^{-1})^{ij}\eta_j)\,.$$

Use this result to obtain an expression for normalized expectation value

$$\langle \bar{\theta}^{i_1} \cdots \bar{\theta}^{i_r}\, \theta^{j_1} \cdots \theta^{j_s}\rangle := \frac{1}{Z(0,0)} \int \mathrm{d}^n\theta\, \mathrm{d}^n\bar{\theta}\ \exp\left(-\bar{\theta}^i B_{ij}\theta^j\right) \bar{\theta}^{i_1} \cdots \bar{\theta}^{i_r}\, \theta^{j_1} \cdots \theta^{j_s}$$

and show that it vanishes whenever $r \neq s$ and whenever $r, s > n$. Interpret your answer in terms of Feynman graphs. [*The result is Wick's theorem for this fermionic theory.*]

2. Consider the partition function

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4} \qquad (\dagger)$$

for a zero–dimensional QFT with a quartic interaction with $\lambda > 0$.

(a) By expanding the integral in λ obtain the n^{th} order perturbative expansion

$$\mathcal{Z}_n(\lambda) = \sum_{\ell=0}^{n} \left(-\frac{\lambda}{4!}\right)^{\ell} \frac{(4\ell)!}{4^\ell\, (2\ell)!\, \ell!}$$

and show for $\ell \leq 3$ that the coefficients a_ℓ of λ^ℓ in this expression are the sums of automorphism factors of the relevant loop Feynman graphs. (At two loops there is only one graph, at three loops there are two graphs and at four loops there are four.)

(b) *Optional but instructive:* Using any computer package, plot $\mathcal{Z}_n(\lambda = \frac{1}{10})$ against n to see that there is a region in n where $\mathcal{Z}_n$ appears to converge, before blowing up as n is increased.

(c) Show that the minimum value of $a_\ell \lambda^\ell$ occurs when $\ell \approx \frac{3}{2\lambda}$. Hence show that the Borel transform $\mathcal{B}\mathcal{Z}(\lambda) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\, a_\ell\, \lambda^\ell$ converges provided $|\lambda| < \frac{3}{2}$ and that in this case

$$\mathcal{Z}(\lambda) = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-z}\, \mathcal{B}Z(z\lambda)$$

so that $\mathcal{Z}(\lambda)$ may be recovered from its Borel transform.

(d) By expanding $\mathrm{e}^{-\frac{1}{2}x^2}$ in the integral (†) obtain the strong coupling expansion

$$\mathcal{Z}(\lambda) = \frac{1}{2\sqrt{\pi}} \sum_{L=0}^{\infty} \frac{(-1)^L}{L!}\, \Gamma\!\left(\frac{L}{2} + \frac{1}{4}\right) \left(\frac{6}{\lambda}\right)^{\frac{L}{2}+\frac{1}{4}}$$

for $\mathcal{Z}(\lambda)$ as a series in $1/\sqrt{\lambda}$. For $\lambda = \frac{1}{10}$ how many terms does one need to obtain the value at which the weak coupling expansion appeared to converge?

3. Consider the action $S(\phi, \psi, \bar{\psi}) = \frac{m^2}{2}\phi^2 + \frac{M}{2}\bar{\psi}\psi + \lambda\phi^2\bar{\psi}\psi$ describing a real bosonic variable ϕ coupled to two independent fermionic (Grassmann) variables ψ, $\bar{\psi}$.

   (a) Treating this as a zero–dimensional QFT, write down the Feynman rules for the propagators and the interaction.

   (b) Integrate out the fermions to show that the effective action for ϕ has an infinite expansion

   $$S_{\text{eff}}(\phi) = \sum_{n=0}^{\infty} \frac{\lambda_n}{n!}\phi^n$$

   in terms of an infinite series of couplings λ_n whose values you should find. Which correlation functions can be computed using S_{eff}?

   (c) Working to order λ^2, compute $\frac{1}{2}\langle\phi^2\rangle$ first by using the original action and then by using the effective action S_{eff}. Check your results agree.

4. Let M be an $N \times N$ Hermitian matrix and consider the integral

   $$\mathcal{Z}(a; N) = \int \mathrm{d}^{2N}\mathrm{M}\, \exp\left(-\frac{1}{2}\mathrm{tr}(\mathrm{M}^2) - \frac{a}{N}\mathrm{tr}(\mathrm{M}^4)\right)$$

   where a is a coupling constant. The measure $\mathrm{d}^{2N}\mathrm{M}$ represents an integral over the real and imaginary parts of each entry of M.

   (a) Represent the propagator as a "double line" where one line edge represents the rows and the other edge represents the columns of M. What are the Feynman rules for this action?

(b) Show that $\mathcal{Z}(a;N)/\mathcal{Z}(0;N)$ can be reduced to an integral over the eigenvalues $\{\lambda_i\}$ of M. [*You may use without proof that the measure* $\mathrm{d}^{2N}\mathrm{M}$ *is invariant under* $\mathrm{M} \to U^{-1}\mathrm{M}U$ *for any unitary matrix* U.]

(c) Show that $\mathcal{Z}(a;N)/\mathcal{Z}(0;N)$ admits a perturbative expansion of the form

$$\ln \frac{\mathcal{Z}(a;N)}{\mathcal{Z}(0;N)} = \sum_{g=0}^{\infty} N^{2-2g} \left(\sum_{n=0}^{\infty} (-a)^n F_{g,n} \right)$$

where $F_{g,n}$ is a combinatoric number, independent of N and a. (You are not required to find an explicit expression for $F_{g,n}$.)

(d) (*) Show that $F_{g,n}$ may be interpreted as the number of ways to cover a genus g Riemann surface with n squares. [*For help with this part of the question, you may wish to consult the first few sections of D. Bessis, C. Itzykson & B. Zuber,* Quantum Field Theory Techniques in Graphical Enumeration, *Adv. Applied Maths* **1**, *109-157, (1980).*]

5. Consider the Quantum Mechanics of a particle moving on $\mathbb{R}^n$ with Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n, \mathrm{d}^n x)$. Obtain (Euclidean time) path integral expressions for the following Heisenberg picture transition functions:

   (a) $\mathrm{Tr}_{\mathcal{H}} \left(P\, \mathrm{e}^{-TH} \right)$, where P is the parity operator $P : \hat{x}^a \to -\hat{x}^a$.

   (b) $\langle \psi_f | \mathrm{e}^{-TH} | \psi_i \rangle$, where $\psi_{i,f}(x) = \langle x | \psi_{i,f} \rangle$ are arbitrary states in the Hilbert space.

   where T is the proper time on the wordline.

   Suppose $n = 1$ and the worldline action includes the potential term $\frac{1}{2} m^2 \omega^2 x^2$. Given that the heat kernel for the (Euclidean) harmonic oscillator is

$$\langle x | \mathrm{e}^{-TH} | y \rangle = \sqrt{\frac{m\omega}{2\pi \, \sin \omega t}} \, \exp \left(-m\omega \frac{(x^2 + y^2) \cos \omega t - 2xy}{2 \sin \omega t} \right) ,$$

   evaluate your expressions for (a) and (b) explicitly in the case that $|\psi_{i,f}\rangle$ are the ground state of the harmonic oscillator. Check that they agree with what you expect from QM, working directly in the energy basis.

6. In a 1d QFT, let x be a real bosonic scalar field and let ψ and $\bar{\psi}$ be fermionic fields. Consider the action

$$S = \int \mathrm{d}\tau \, \left[\frac{1}{2} \left(\frac{\partial x}{\partial \tau} \right)^2 + \bar{\psi} \frac{\partial \psi}{\partial \tau} + \frac{1}{2} \lambda^2 h'(x)^2 + \lambda \, h''(x) \, \bar{\psi} \psi \right]$$

   where λ is a coupling constant and $h(x)$ is a smooth function of $x(\tau)$ (and h' the derivative of this function).

(a) Show that this action is invariant under the transformations

$$\begin{aligned}\delta x &= \epsilon\bar{\psi} - \bar{\epsilon}\psi \\ \delta\psi &= \epsilon(-\dot{x} + \lambda\, h'(x)) \\ \delta\bar{\psi} &= \bar{\epsilon}(\dot{x} + \lambda\, h'(x))\end{aligned}$$

where ϵ and $\bar{\epsilon}$ are constant fermionic parameters and $\dot{x} = \partial x/\partial\tau$. Find the conserved charges Q and $\bar{Q}$ associated to these two symmetries.

(b) Show that $Q\bar{Q} + \bar{Q}Q = 2H$ where H is the Hamiltonian associated to the above Lagrangian. Hence or otherwise show that the energies of the system are non–negative, and that the ground state $|\Psi\rangle$ obeys $Q|\Psi\rangle = \bar{Q}|\Psi\rangle = 0$.

(c) Put this theory on a circle and take $x(\tau)$, $\psi(\tau)$ and $\bar{\psi}(\tau)$ each to be periodic. Show that the partition function $\mathcal{Z} = \int \mathcal{D}x\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\ \mathrm{e}^{-S}$ with these boundary conditions is independent both of the radius of the circle and of λ, provided $\lambda \neq 0$.

(d) What is the operator expression to which this partition function corresponds? [*Hint: Think carefully about the fermionic boundary conditions.*]

(e) (∗) Argue that the only field configurations that contribute to $\mathcal{Z}$ have $\dot{x} = h'(x) = 0$. By expanding the action in a neighbourhood of these configurations, evaluate $\mathcal{Z}$ in the case that $h(x)$ is a polynomial of degree n with isolated roots.

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Advanced Quantum Field Theory
Example Sheet 2

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (∗) may be more difficult.

1. Furry's theorem states that $\langle \tilde{A}_{\mu_1}(k_1)\cdots \tilde{A}_{\mu_n}(k_n)\rangle = 0$ when n is odd, where $\tilde{A}(k)$ is the photon field in momentum space. It is a consequence of charge conjugation invariance.
   (a) In scalar QED, charge conjugation swaps ϕ and $\bar{\phi}$. How must the photon field A_μ transform if the action is to be invariant?
   (b) Prove Furry's theorem in scalar QED using the path integral.
   (c) Does Furry's theorem hold for off–shell photons with $k_\mu k^\mu \neq 0$?

2. In the Lorenz gauge $\partial^\mu A_\mu = 0$, the classical equation of motion for the photon that follows from the QED action is $\Box A_\mu = e j_\mu = e\bar{\psi}\gamma_\mu\psi$.
   (a) Obtain the Dyson–Schwinger equation
   $$\Box^{(x)}\langle A_\mu(x)A_\nu(y)\psi(x_1)\bar{\psi}(x_2)\rangle = e\langle j_\mu(x)A_\nu(y)\psi(x_1)\bar{\psi}(x_2)\rangle - \delta^4(x-y)\,\delta_{\mu\nu}\langle\psi(x_1)\bar{\psi}(x_2)\rangle$$
   for the photon in the quantum theory. What is the corresponding equation for the electron?
   (b) Use the Dyson–Schwinger equation to related the QED correlation function $\langle j^\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle$ to the exact electron–photon vertex function.
   (c) Assuming the path integral measure is invariant, show that the Ward identity
   $$(k_1 - k_2)_\mu \Gamma^\mu(k_1,k_2) = \mathrm{i}S^{-1}(k_2) - \mathrm{i}S^{-1}(k_1) \qquad (0.1)$$
   obtained in lectures implies that the momentum–space three-point function $\langle \tilde{A}_\mu(p)\psi(k_1)\bar{\psi}(k_2)\rangle$ obeys
   $$p^\mu\langle \tilde{A}_\mu(p)\psi(k_1)\bar{\psi}(k_2)\rangle = 0$$
   when the electrons are on–shell, whether or not $p^2 = 0$. What is the significance of this result?

3. Suppose that an action involving N real scalar fields ϕ^i is invariant under global $\mathrm{SO}(N)$ rotations of the different scalars. Show that the corresponding charges obey the $\mathfrak{so}(N)$ algebra $[\hat{Q}_a, \hat{Q}_b] = \mathrm{i} f_{ab}^c \hat{Q}_c$.

4. Show that under the redefinition $g_i \to g_i'(g_j)$ of the couplings of a theory at scale Λ, the β-functions transform as
$$\beta_i \to \beta_i' = \frac{\partial g_i'}{\partial g_j}\beta_j\,.$$
Show that in a theory with a single coupling g, the first two terms in the β–function $\beta(g) = ag^3 + cg^5 + \mathcal{O}(g^7)$ are invariant under any coupling constant redefinition of the form $g \to g' = g + \mathcal{O}(g^3)$. Show that it is possible to choose this redefinition so as to remove all terms *except* these first two.

5. A certain theory has a single coupling g, with β-function $\beta(g) = -ag^3 - bg^5 + \mathcal{O}(g^7)$. Solve the Callan–Symanzik equation for the running coupling to find
$$\frac{1}{g^2} = a\ln\frac{\Lambda^2}{\mu^2} + \frac{b}{a}\ln\left(\ln\frac{\Lambda^2}{\mu^2}\right) + \mathcal{O}(1/\ln(\Lambda^2/\mu^2))\,,$$
or equivalently
$$\mu^2 = \Lambda^2\,\mathrm{e}^{-\frac{1}{ag^2}}\,(ag^2)^{-\frac{b}{a}}\left(1+\mathcal{O}(g^2)\right)\,.$$
Now suppose that $g' = g + cg^3 + \mathcal{O}(g^5)$. Show that ${g'}^2$ can be expresssed in terms of μ'^2 as above, where $\ln({\mu'}^2/\mu^2) = c/b$. What is the significance of this calculation?

6. Let ψ^i denote a (fermionic) Dirac spinor field transforming in the fundamental representation of a $\mathrm{U}(N)$ gauge group, and let $\bar{\psi}_j$ denote the Dirac conjugate spinor transforming in the antifundamental. Let ${(A_\mu)^i}_j$ denote the gauge field for this interaction. Write down all possible $\mathrm{U}(N)$ gauge invariant local operators involving these fields that are relevant or marginal near the Gaussian critical point, in the cases that the space–time has dimension $d = 4$, $d = 3$ and $d = 2$.

7. Consider a four dimensional theory whose only couplings are a mass parameter m^2 and a marginally relevant coupling g.

   (a) Write down generic expressions for the β-functions in such a theory to lowest non–trivial order. (You should be able to identify the *values* of the classical contributions to the β-functions, and the *sign* of the leading–order quantum correction to $\beta(g)$.)

   (b) Sketch the RG flows for this theory.

   (c) Suppose that $g(\Lambda') = 0.1$ when the cut–off Λ' is fixed at 10^5 GeV. If $m^2(\Lambda')$ is measured to be 100 GeV, what value of $m^2(\Lambda)$ would be needed at the higher scale $\Lambda = 10^{19}$ GeV?

(d) Suppose you changed your value of $m^2(\Lambda)$ by one part in 10^{20}. What would be the change in $m^2(\Lambda')$?

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Advanced Quantum Field Theory
Example Sheet 3

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (∗) may be more difficult.

1. Consider the theory given by the action

$$S[\phi] = \int \mathrm{d}^d x\, \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 \ .$$

where ϕ is a real scalar field.

(a) Determine all connected one loop graphs, complete with their appropriate symmetry factors, which contribute to

$$\langle\phi(x)\phi(y)\rangle\,, \qquad \langle\phi(x)\phi(y)\phi(z)\rangle \qquad \text{and} \qquad \langle\phi(x)\phi(y)\phi(z)\phi(w)\rangle\,,$$

expressing your answer in terms of integrals over d-dimensional loop momenta. [You are not required to evaluate the integrals.]

(b) Now set $\lambda = 0$ so that just the cubic interaction remains. Determine the momentum space correlation function $\int\prod_{i=1}^3 \mathrm{d}^d x_i\, \mathrm{e}^{\mathrm{i}p_i\cdot x_i}\,\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle$ to one loop accuracy.

2. Consider a (neutral) scalar field ϕ with potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{6}\mu^{\epsilon/2}g(\mu)\phi^3$ in dimension $d = 6-\epsilon$. Here μ is an arbitrary mass scale introduced so that the coupling $g(\mu)$ is dimensionless.

(a) Draw the one-loop one particle irreducible graph which contributes to the propagator at order g^2.

(b) Using dimensional regularisation, show that the divergent part of the corresponding integral for the six dimensional theory is

$$-\frac{1}{\epsilon}\frac{g^2}{(4\pi)^3}\left(m^2 + \frac{1}{6}p^2\right)\,,$$

where p is the external momentum. Also compute the divergence corresponding to the one particle irreducible one-loop graph that gives a g^3 correction to three point function, and find the one loop divergence for the one point function.

(c) Show that in six dimensions all these divergences may be cancelled by introducing the counterterm action

$$S_{\rm ct}[\phi] = \int \mathrm{d}^d x \, \mathcal{L}_{\rm ct} := \frac{1}{\epsilon}\frac{1}{6(4\pi)^3}\left[\int \mathrm{d}^d x \, \frac{1}{2}g^2(\partial\phi)^2 + \mu^{-\epsilon}V''(\phi)^3\right].$$

Check that $\mathcal{L}_{\rm ct}$ has dimension d.

(d) Determine the β-function for the coupling g and show that $\beta(g) < 0$ at small g. Does the theory have a continuum limit in perturbation theory? Do you expect this to survive non–perturbatively?

3. Scalar QED describes the interactions of a photon with a complex scalar field. In d dimensions it is defined by the action

$$S[A,\phi] = \int \mathrm{d}^d x \, \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(D^\mu\phi)^* \, D_\mu\phi + \frac{m^2}{2}\phi^*\phi$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi = \partial_\mu\phi + \mathrm{i}eA_\mu\phi$.

(a) Show that, not including counterterms, there are two distinct 1-loop Feynman graphs that contribute to vacuum polarization in scalar QED. One of these diagrams leads to an integral that is independent of the external momentum. What is its role?

(b) By considering vacuum polarization, show that when $d = 4$, the 1-loop β-function for the dimensionless coupling g corresponding to the charge e is

$$\beta(g) = \frac{g^3}{48\pi^2}$$

in the $\overline{\rm MS}$ scheme. How does the theory behave at scales far below the mass of the scalar?

4. Consider the theory of a scalar field $\phi(x)$ and fermionic Dirac field ψ, with action

$$S[\phi,\psi] = \int \mathrm{d}^4 x \, \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \bar\psi(\mathrm{i}\partial\!\!\!/ + M)\psi + g\,\phi\bar\psi\gamma_5\psi + \frac{\lambda}{4!}\phi^4$$

in four dimensional Euclidean space, where $\gamma_5 = +\gamma_1\gamma_2\gamma_3\gamma_4$ is the product of Dirac matrices obeying $\{\gamma^\mu,\gamma^\nu\} = 2\,\delta^{\mu\nu}$ in Euclidean signature.

(a) Show that $(\gamma_5)^2 = +1$ and that $\{\gamma_5,\gamma^\mu\} = 0$. Hence show that the action is invariant under the global transformation

$$\phi \to -\phi \qquad\qquad \psi \to \mathrm{e}^{-\mathrm{i}\pi\gamma_5/2}\psi\,.$$

Assuming that the path integral measure is also invariant under this transformation, show that renormalization cannot generate any vertices involving odd powers of the scalar field unless they are accompanied by an odd power of $\bar\psi\gamma_5\psi$, as in the original action.

(b) What counterterms are necessary when studying the continuum limit of this theory? Using dimensional regularization and the on-shell renormalization scheme, evaluate these counterterms to 1-loop accuracy, and show that the physical amplitudes are finite.

5. Consider Yang–Mills theory with a semi–simple gauge group G, minimally coupled to a Dirac spinor transforming in the fundamental representation.

   (a) How do the gauge field and fermion transform under a *constant* gauge transformation $h \in G$?

   (b) Obtain an expression for the Noether current J_μ^{Noether} for this transformation, and show that it obeys the 'naive' conservation equation $\partial^\mu J_\mu^{\text{Noether}} = 0$.

   (c) Using the equations of motion, show that the corresponding Noether charge reduces to an integral over a $(d-2)$-dimensional surface at infinity. What is the significance of this fact?

6. (*) In General Relativity, the analogue of gauge transformations are diffeomorphisms of a space–time M. Show that there are no local observables in Quantum Gravity. So what do experimentalists measure?

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Advanced Quantum Field Theory
Example Sheet 4

Please email me with any comments about these problems, particularly if you spot an error. Questions marked with an asterisk may be more challenging.

1. Let t_A be the generators of a Lie algebra $\mathfrak{g}$, $[t_A, t_B] = \mathrm{i} f_{AB}^C t_C$, and let c^A be anticommuting variables. Show that

$$Q := c^A t_A - \frac{1}{2} f_{BC}^A c^B c^C \frac{\partial}{\partial c^A}$$

satisfies $Q^2 = 0$. Suppose $t_A = 0$ and also that $f_{ABC} = k_{CD} f_{AB}^D$ is completely antisymmetric, where k_{CD} is the Killing form on $\mathfrak{g}$. If $X = f_{ABC}\, c^A c^B c^C$ show that $QX = 0$ but that $X \neq QY$.

2. Consider a gauge-fixed action for a free (Abelian) gauge field A_μ of the form

$$S = \int \mathrm{d}^D x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + h\, \partial^\mu A_\mu + \frac{\xi}{2} h^2 + \bar{c}\, \partial^2 c \right)$$

where h is an auxiliary bosonic field and $(c, \bar{c})$ are anticommuting ghost and antighost fields.

   (a) Verify that this action is invariant under the BRST transformations $\delta A_\mu = \epsilon \partial_\mu c$, $\delta c = 0$, $\delta \bar{c} = -\epsilon h$, $\delta h = 0$ and that δ is nilpotent.

   (b) Show that the action can be written in the form

$$S = -\int \mathrm{d}^D x \left(\frac{1}{2} \Phi^{\mathrm{T}} \Delta \Phi + \bar{c}(-\partial^2) c \right)$$

   where $\Phi = \begin{pmatrix} A_\mu \\ h \end{pmatrix}$, Φ^{T} is its transpose and where

$$\Delta = \begin{pmatrix} -\partial^2 \delta^\mu{}_\nu + \partial^\mu \partial_\nu & \partial_\nu \\ -\partial^\mu & -\xi \end{pmatrix} .$$

(c) Obtain equations for the normalized correlation functions $\langle \Phi(x)\, \Phi(0)^{\mathrm{T}} \rangle$ and $\langle c(x)\, \bar{c}(0) \rangle$ and show that

$$\int \mathrm{d}^D x\, \mathrm{e}^{-\mathrm{i}p\cdot x} \langle \Phi(x)\, \Phi(0)^{\mathrm{T}} \rangle = -\frac{\mathrm{i}}{p^2} \begin{pmatrix} \delta_\mu{}^\nu - (1-\xi)\frac{p_\mu p^\nu}{p^2} & \mathrm{i}p_\mu \\ -\mathrm{i}p^\nu & 0 \end{pmatrix}$$

$$\int \mathrm{d}^D x\, \mathrm{e}^{-\mathrm{i}p\cdot x} \langle c(x)\, \bar{c}(0) \rangle = -\frac{\mathrm{i}}{p^2} \, .$$

(d) Assuming that $\langle \delta Y \rangle = 0$ for any Y, consider $\langle \delta \left(\Phi(x)\, \bar{c}(0) \right) \rangle$ and show that we must have $\langle h(x)\, h(0) \rangle = 0$. Obtain also a relation between $\langle c(x)\, \bar{c}(0) \rangle$ and $\langle A_\mu(x)\, h(0) \rangle$ which should be verified.

3. For a gauge theory coupled to scalars the single particle states are

$$|A^A_\mu(p)\rangle \, , \qquad |\phi_i(p)\rangle \, , \qquad |c^A(p)\rangle \, , \qquad |\bar{c}^A(p)\rangle \, ,$$

where A runs over a basis of the adjoint representation and i similarly indexes the R representation. These states have non-zero scalar products

$$\langle A^A_\mu(p)|A^B_\nu(p')\rangle = \eta_{\mu\nu}\delta^{AB}\delta_{pp'} \qquad \langle \phi_i(p)|\phi_j(p')\rangle = \delta_{ij}\delta_{pp'}$$
$$\langle c^A(p)|\bar{c}^B(p')\rangle = \langle \bar{c}^A(p)|c^B(p')\rangle = \delta^{AB}\delta_{pp'}$$

where $\delta_{pp'} := (2\pi)^{D-1}\, 2p^0\, \delta^{(D-1)}(\mathbf{p}-\mathbf{p}')$. The non-zero action of the BRST charge Q is given by

$$Q|A^A_\mu(p)\rangle = \alpha p_\mu |c^A(p)\rangle \, , \quad Q|\phi_i(p)\rangle = \sum_A v_{iA} |c^A(p)\rangle$$
$$Q|\bar{c}^A(p)\rangle = \beta p^\mu |A^A_\mu(p)\rangle + \sum_i \bar{v}_{Ai} |\phi_i(p)\rangle$$

while the ghost charge Q_{gh} acts non-trivially as

$$Q_{\mathrm{gh}}|c^A(p)\rangle = \mathrm{i}|c^A(p)\rangle \, , \qquad Q_{\mathrm{gh}}|\bar{c}^A(p)\rangle = -\mathrm{i}|\bar{c}^A(p)\rangle \, .$$

Verify that this is compatible with Q and Q_{gh} being Hermitian if α, β, v_{iA} and $\bar{v}_{Ai}$ are related appropriately. Assume a basis has been chosen so that $\sum_i \bar{v}_{Ai} v_{iB} = \delta_{AB}\rho_A$ and so is diagonal. Find the conditions under which the BRST charge $Q^2 = 0$. Use this to determine the possible physical single particle states.

4. Consider a gauge invariant Lagrangian density of the form

$$\mathcal{L}(A) = -\frac{1}{4} \mathrm{tr} \left(F^{\mu\nu} X(D^2) F_{\mu\nu} \right) \, ,$$

where $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}]$ and where $X(D^2) = 1 + (-D^2)^r/\Lambda^{2r}$ for some scale Λ. The full quantum Lagrangian with gauge fixing and ghost fields is

$$\mathcal{L}_q(A, c, \bar{c}) = \mathcal{L}(A) - \frac{1}{2\xi} \mathrm{tr} \left(\partial^\mu A_\mu\, X(\partial^2)\, \partial^\nu A_\nu \right) + \mathrm{tr} \left(\bar{c}\, X(\partial^2)\, \partial^\mu D_\mu c \right) \, .$$

(a) Show that the Feynman rules require that the gauge and ghost propagators are each proportional to p^{-2-2r}.

(b) Show that there must be vertices with n gauge field legs $n = 3, \ldots 2r+4$ and with $2r+4-n$ powers of momentum, but that there is just a single vertex involving both ghost and gauge fields with $2r+1$ momentum factors.

(c) Hence show that, in four dimensions, the superficial degree of divergence of an ℓ-loop Feynman graph with E_A external gauge field lines and $E_{\rm gh}$ ghost field lines is

$$\delta = 4 - E_A - E_{\rm gh} - 2r(\ell - 1)\,.$$

5. *Consider pure (= no charged matter) electrodynamics with Lagrangian $\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu}$. Let $\mathrm{W}_\gamma[A] := \exp\left(\mathrm{i}\oint_\gamma A_\mu\, \mathrm{d}x^\mu\right)$ be a Wilson loop around a closed curve γ.

(a) Show that

$$\langle \mathrm{W}_\gamma[A]\rangle = \exp\left[-\frac{e^2}{8\pi^2}\oint_\gamma \mathrm{d}x^\mu \oint_\gamma \mathrm{d}y_\mu\, \frac{1}{(x-y)^2}\right]\,.$$

(b) Now suppose γ is a large rectangle with space-like width L and time-like length T. Compute $\langle \mathrm{W}_\gamma[A]\rangle$ in the limit $T \gg L$. By comparing your result to the usual expression for time evolution, show that the potential between two point-like charges at fixed separation L in electrodynamics is $V(L) = -e^2/4\pi L$.

(c) In Feynman gauge, the propagator for a non-Abelian gauge field is

$$\langle A^B_\mu(x) A^C_\nu(y)\rangle = \mathrm{i}\eta_{\mu\nu}\,\delta^{BC}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2}\,.$$

Compute the expectation value of a Wilson loop in pure SU(N) Yang-Mills theory to lowest non-trivial order in the coupling g^2. [*Your result should depend on a choice of the representation R of $\mathfrak{su}(N)$.*]

(d) Show that, to this order, the Coulomb potential of non-Abelian gauge theory is $V(L) = -g^2 C_2(R)/4\pi L^2$, where $C_2(R)$ is the quadratic Casimir of the R representation.