Part III — Advanced Quantum Field Theory
Definitions

Based on lectures by D. B. Skinner
Notes taken by Dexter Chua

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course builds on the Michaelmas Quantum Field Theory course, using techniques of path integrals and functional methods to study quantum gauge theories. Gauge Theories are a generalisation of electrodynamics and form the backbone of the Standard Model — our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie Group at every point in space-time and yet still describe the same physics. Quantising a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry. We discuss the construction of gauge theories and their most important observables, Wilson Loops. Time permitting, we will explore the possibility that a classical symmetry may be broken by quantum effects. Such anomalies have many important consequences, from constraints on interactions between matter and gauge fields, to the ability to actually render a QFT inconsistent.

A further major component of the course is to study Renormalization. Wilson's picture of Renormalisation is one of the deepest insights into QFT — it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the Renormalisation Group (RG) flow. The course explores renormalisation systematically, from the use of dimensional regularisation in perturbative loop integrals, to the difficulties inherent in trying to construct a quantum field theory of gravity. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a non-Abelian gauge theory can effectively become small at high energies. Known as "asymptotic freedom", this phenomenon revolutionised our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrise possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Pre-requisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be very helpful.

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1 QFT in zero dimensions

1.1 Free theories

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2 QFT in one dimension (i.e. QM)

2.1 Quantum mechanics

Definition (Local operator)**.** A *local operator* $\mathcal{O}(t)$ is one which depends on the values of the fields and finitely many derivatives just at one point $t \in M$.

Definition (Lebesgue measure)**.** A *Lebesgue measure* $\mathrm{d}\mu$ on an inner product space V obeys the following properties

- For all non-empty open subsets $U \subseteq \mathbb{R}^D$, we have

$$\mathrm{vol}(U) = \int_U \mathrm{d}\mu > 0.$$

- If U' is obtained by translating U, then

$$\mathrm{vol}(U') = \mathrm{vol}(U).$$

- Every $x \in V$ is contained in at least one open neighbourhood U_x with finite volume.

2.2 Feynman rules

2.3 Effective quantum field theory

2.4 Quantum gravity in one dimension

3 Symmetries of the path integral

3.1 Ward identities

3.2 The Ward–Takahashi identity

Definition (Exact propagator)**.** The *exact (electron) propagator* is defined by

$$S(k) = \int \mathrm{d}^4 y\; e^{ik\cdot y}\langle \psi(y)\bar{\psi}(0)\rangle,$$

evaluated in the full, interacting theory.

Definition (One-particle irreducible graph)**.** A *one-particle irreducible graph* for $\langle\psi\bar{\psi}\rangle$ is a connected Feynman diagram (in momentum space) with two external vertices $\bar{\psi}$ and ψ such that the graph cannot be disconnected by the removal of one internal line.

Definition (Exact electromagnetic vertex)**.** The *exact electromagnetic vertex* $\Gamma_\mu(k_1, k_2)$ is defined by

$$\begin{aligned}\delta^4(p + k_1 - k_2)&S(k_1)\Gamma_\mu(k_1, k_2)S(k_2)\\ &= \int \mathrm{d}^4 x\; \mathrm{d}^4 x_1\; \mathrm{d}^4 x_2\; \langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle e^{ip\cdot x}e^{ik_1\cdot x_1}e^{-ik_2\cdot x_2}.\end{aligned}$$

4 Wilsonian renormalization

4.1 Background setting

4.2 Integrating out modes

Definition (Beta function)**.** The *beta function* of the coupling g_i is

$$\beta_i(g_j) = \Lambda \frac{\partial g_i}{\partial \Lambda}.$$

Definition (Anomalous dimension)**.** The *anomalous dimension* of ϕ by

$$\gamma_\phi = -\frac{1}{2}\Lambda \frac{\partial \log Z_\Lambda}{\partial \Lambda}$$

4.3 Correlation functions and anomalous dimensions

Definition ($\Gamma_\Lambda^{(n)}$)**.** We write

$$\Gamma_\Lambda^{(n)}(\{x_i\}, g_i) = \frac{1}{\mathcal{Z}} \int_{\leq \Lambda} \mathrm{D}\phi\, e^{-S_\Lambda[\phi, g_i]} \phi(x_1) \cdots \phi(x_n) = \langle \varphi(x_1) \cdots \varphi(x_n) \rangle.$$

4.4 Renormalization group flow

Definition (Critical point)**.** A *critical point* is a point in the configuration space, i.e. a choice of couplings $g_i = g_i^*$ such that $\beta_i(g_i^*) = 0$.

4.5 Taking the continuum limit

4.6 Calculating RG evolution

5 Perturbative renormalization

5.1 Cutoff regularization

5.2 Dimensional regularization

5.3 Renormalization of the ϕ^4 coupling

5.4 Renormalization of QED

6 Non-abelian gauge theory

6.1 Bundles, connections and curvature

Definition (Section)**.** Let $p : E \to M$ be a map between manifolds. A *section* is a smooth map $s : M \to E$ such that $p \circ s = \mathrm{id}_M$.

Definition (Vector bundle)**.** Let M be a manifold, and V a vector space. A *vector bundle* over M with *typical fiber* V is a manifold E with a map $\pi : E \to M$ such that for all $x \in M$, the fiber $E_x \cong \pi^{-1}(\{x\})$ is a vector space that is isomorphic to V.

Moreover, we require that for each $x \in M$, there exists an open neighbourhood U of x, and a diffeomorphism $\Phi : U \times V \to \pi^{-1}(U)$ such that $\pi(\Phi(y, v)) = y$ for all y, and $\Phi(y, \cdot\,) : \{y\} \times V \to E_y$ is a *linear isomorphism* of vector spaces.

Such a Φ is called a *local trivialization* of E, and U is called a *trivializing neighbourhood*.

Notation. Let $E \to M$ be a vector bundle. Then we write $\Omega^0_M(E)$ for the vector space of sections of $E \to M$. Locally, we can write an element of this as X^a, for $a = 1, \cdots, \dim E_x$.

More generally, we write $\Omega^p_M(E)$ for sections of $E \otimes \bigwedge^p T^*M \to M$, where $\bigwedge^p T^*M$ is the bundle of p-forms on M. Elements can locally be written as $X^a_{\mu_1 \ldots \mu_n}$.

If V is a vector space, then $\Omega^p_M(V)$ is a shorthand for $\Omega^p_M(V \times M)$.

Definition (G-bundle)**.** Let V be a vector space, and $G \leq \mathrm{GL}(V)$ be a Lie subgroup. Then a G-bundle over M is a vector bundle over M with fiber V, equipped with a trivializing cover such that the transition functions take value in G.

Definition (G-bundle)**.** Let V be a representation, G a Lie group, and $\rho : G \to \mathrm{GL}(V)$ a representation. Then a G-bundle consists of the following data:

(i) A vector bundle $E \to M$.

(ii) A trivializing cover $\{U_\alpha\}$ with transition functions $t_{\alpha\beta}$.

(iii) A collection of maps $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying the cocycle conditions such that $t_{\alpha\beta} = \rho \circ \varphi_{\alpha\beta}$.

Definition (Principal G-bundle)**.** Let G be a Lie group, and M a manifold. A *principal G-bundle* is a map $\pi : P \to M$ such that $\pi^{-1}(\{x\}) \cong G$ for each $x \in M$. Moreover, $\pi : P \to M$ is locally trivial, i.e. it locally looks like $U \times G$, and transition functions are given by left-multiplication by an element of G.

More precisely, we are given an open cover $\{U_\alpha\}$ of M and diffeomorphisms

$$\Phi_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha)$$

satisfying $\pi(\Phi_\alpha(x, g)) = x$, such that the transition functions

$$\Phi_\alpha^{-1} \circ \Phi_\beta : (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G$$

is of the form

$$(x, g) \mapsto (x, t_{\alpha\beta}(x) \cdot g)$$

for some $t_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

Definition (Connection)**.** A *connection* is a linear map $\nabla : \Omega_M^0(E) \to \Omega_M^1(E)$ satisfying

(i) Linearity:
$$\nabla(\alpha_1 s_1 + \alpha_2 s_2) = \alpha_1(\nabla s_1) + \alpha_2(\nabla s_2)$$
for all $s_1, s_2 \in \Omega_M^0(E)$ and α_1, α_2 constants.

(ii) Leibnitz property:
$$\nabla(fs) = (\mathrm{d}f)s + f(\nabla S)$$
for all $s \in \Omega_M^0(E)$ and $f \in C^\infty(M)$, where, $\mathrm{d}f$ is the usual exterior derivative of a function, given in local coordinates by
$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu.$$

Given a vector field V on M, the *covariant derivative* of a section in the direction of V is the map
$$\nabla_V : \Omega_M^0(E) \to \Omega_M^0(E)$$
defined by
$$\nabla_V s = V \lrcorner \nabla s = V^\mu \nabla_\mu s.$$

6.2 Yang–Mills theory

6.3 Quantum Yang–Mills theory

6.4 Faddeev–Popov ghosts

6.5 BRST symmetry and cohomology

Definition (BRST operator)**.** The *BRST operator* $\mathcal{Q}$ is defined by
$$\mathcal{Q}A_\mu = \nabla_\mu c \qquad \mathcal{Q}\bar{c} = ih$$
$$\mathcal{Q}c = -\frac{1}{2}[c, c] \qquad \mathcal{Q}h = 0.$$

This extends to an operator on B by sending all constants to 0, and for $f, g \in B$ of definite statistics, we set
$$\mathcal{Q}(fg) = (-1)^{|f|} f\mathcal{Q}g + (\mathcal{Q}f)g, \quad \mathcal{Q}(\partial_\mu f) = \partial_\mu \mathcal{Q}f.$$

In other words, $\mathcal{Q}$ is a *graded derivation*.

Definition (BRST exact)**.** We say $a \in B$ is *BRST exact* if $a = \mathcal{Q}b$ for some $b \in B$.

Definition (BRST closed)**.** We say $a \in B$ is *BRST closed* if $\mathcal{Q}a = 0$.

6.6 Feynman rules for Yang–Mills

6.7 Renormalization of Yang–Mills theory