Part III — Classical and Quantum Solitons
Theorems with proof

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Solitons are solutions of classical field equations with particle-like properties. They are localised in space, have finite energy and are stable against decay into radiation. The stability usually has a topological explanation. After quantisation, they give rise to new particle states in the underlying quantum field theory that are not seen in perturbation theory. We will focus mainly on kink solitons in one space dimension, on gauge theory vortices in two dimensions, and on Skyrmions in three dimensions.

Pre-requisites

This course assumes you have taken Quantum Field Theory and Symmetries, Fields and Particles. The small amount of topology that is needed will be developed during the course.

Reference

N. Manton and P. Sutcliffe, *Topological Solitons*, CUP, 2004

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2.2 Global $U(1)$ Ginzburg–Landau vortices

Lemma. Assume Φ is a smooth solution of the ungauged Ginzburg–Landau equation in some domain. Then at any interior maximum point x_* of $|\Phi|$, we have $|\Phi(x_*)| \leq 1$.

Proof. Consider the function

$$W(x) = 1 - |\Phi(x)|^2.$$

Then we want to show that $W \geq 0$ when W is minimized. We note that if W is at a minimum, then the Hessian matrix must have non-negative eigenvalues. So, taking the trace, we must have $\Delta W(x_*) \geq 0$. Now we can compute ΔW directly. We have

$$\begin{aligned}
\nabla W &= -2(\Phi, \nabla\Phi) \\
\Delta W &= \nabla^2 W \\
&= -2(\Phi, \Delta\Phi) - 2(\nabla\Phi, \nabla\Phi) \\
&= \lambda|\Phi|^2 W - 2|\nabla\Phi|^2.
\end{aligned}$$

Thus, we can rearrange this to say

$$2|\nabla\Phi|^2 + \Delta W = \lambda|\Phi|^2 W.$$

But clearly $2|\nabla\Phi|^2 \geq 0$ everywhere, and we showed that $\Delta W(x_*) \geq 0$. So we must have $W(x_*) \geq 0$. □

2.3 Abelian Higgs/Gauged Ginzburg–Landau vortices

Proposition. If f is a smooth real-valued function, and Φ and Ψ are smooth complex scalar fields, then

$$\begin{aligned}
\mathrm{D}(f\Phi) &= (\mathrm{d}f)\ \Phi + f\ \mathrm{D}\Phi, \\
\mathrm{d}(\Phi, \Psi) &= (\mathrm{D}\Phi, \Psi) + (\Phi, \mathrm{D}\Psi).
\end{aligned}$$

(Here $(\,\cdot\,)$ is the real inner product defined above.) In coordinates, these take the form

$$\begin{aligned}
\mathrm{D}_j(f\Phi) &= (\partial_j f)\ \Phi + f\ \mathrm{D}_j\Phi \\
\partial_j(\Phi, \Psi) &= (\mathrm{D}_j\Phi, \Psi) + (\Phi, \mathrm{D}_j\Psi).
\end{aligned}$$

Proposition. If Φ is a smooth scalar field, then

$$(\mathrm{D}_1\mathrm{D}_2 - \mathrm{D}_2\mathrm{D}_1)\Phi = -iB\Phi.$$

Proposition.

$$\mathrm{D}\mathrm{D}\Phi = -iF\Phi.$$

Proof.

$$\begin{aligned}
\mathrm{D}\mathrm{D}\Phi &= (\mathrm{d} - iA)(\mathrm{d}\Phi - iA\Phi) \\
&= \mathrm{d}^2\Phi - i\mathrm{d}(A\Phi) - iA\,\mathrm{d}\Phi - A \wedge A\,\Phi \\
&= -i\mathrm{d}(A\Phi) - iA\,\mathrm{d}\Phi \\
&= -i\mathrm{d}A\,\Phi + iA\,\mathrm{d}\Phi - iA\,\mathrm{d}\Phi \\
&= -i(\mathrm{d}A)\,\Phi \\
&= -iF\Phi.
\end{aligned}$$

□

Lemma. Assume Φ is a smooth solution of the gauged Ginzburg–Landau equation in some domain. Then at any interior maximum point x_* of $|\Phi|$, we have $|\Phi(x_*)| \leq 1$.

Proof. Consider the function

$$W(x) = 1 - |\Phi(x)|^2.$$

Then we want to show that $W \geq 0$ when W is minimized. We note that if W is at a minimum, then the Hessian matrix must have non-negative eigenvalues. So, taking the trace, we must have $\Delta W(x_*) \geq 0$. Now we can compute ΔW directly. We have

$$\begin{aligned}
\partial_j W &= -2(\Phi, \mathrm{D}_j\Phi) \\
\Delta W &= \partial_j \partial_j W \\
&= -2(\Phi, \mathrm{D}_j\mathrm{D}_j\Phi) - 2(\mathrm{D}_j\Phi, \mathrm{D}_j\Phi) \\
&= \lambda|\Phi|^2 W - 2|\nabla\Phi|^2.
\end{aligned}$$

Thus, we can rearrange this to say

$$2|\nabla\Phi|^2 + \Delta W = \lambda|\Phi|^2 W.$$

But clearly $2|\nabla\Phi|^2 \geq 0$ everywhere, and we showed that $\Delta W(x_*) \geq 0$. So we must have $W(x_*) \geq 0$. □

2.4 Bogomolny/self-dual vortices and Taubes' theorem

Theorem (Taubes' theorem)**.** For $\lambda = 1$, the space of (gauge equivalence classes of) solutions of the Euler–Lagrange equations $\delta V_1 = 0$ with winding number N is $\mathcal{M} \cong \mathbb{C}^N$.

To be precise, given $N \in \mathbb{N}$ and an unordered set of points $\{Z_1, \cdots, Z_N\}$, there exists a smooth solution $A(x; Z_1, \cdots, Z_N)$ and $\Phi(x; Z_1, \cdots, Z_n)$ which solves the Euler-Lagrange equations $\delta V_1 = 0$, and also the so-called *Bogomolny equations*

$$\mathrm{D}_1\Phi + i\mathrm{D}_2\Phi = 0, \quad B = \frac{1}{2}(1 - |\Phi|^2).$$

Moreover, Φ has exactly N zeroes $Z_1, \cdots, Z_N$ counted with multiplicity, where (using the complex coordinates $z = x_1 + ix_2$)

$$\Phi(x; Z_1, \cdots, Z_N) \sim c_j(z - Z_j)^{n_j}$$

as $z \to Z_j$, where $n_j = |\{k : Z_k = Z_j\}|$ is the multiplicity and c_j is a nonzero complex number.

This is the unique such solution up to gauge equivalence. Furthermore,

$$V_1(A(\,\cdot\,, Z_1, \cdots, Z_N), \Phi(\,\cdot\,; Z_1, \cdots, Z_N)) = \pi N \tag{$*$}$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} B \, \mathrm{d}^2 x = N = \text{winding number}.$$

Finally, this gives all finite energy solutions of the gauged Ginzburg–Landau equations.

Lemma. We have

$$V_1(A, \Phi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\left(B - \frac{1}{2}(1 - |\Phi|^2)\right)^2 + 4|\bar{\partial}_A \Phi|^2\right) \, \mathrm{d}^2 x + \pi N,$$

where

$$\bar{\partial}_A \Phi = \frac{1}{2}(\mathrm{D}_1 \Phi + i \mathrm{D}_2 \Phi).$$

Proof. We complete the square and obtain

$$V_1(A, \Phi) = \frac{1}{2}\int \left(\left(B - \frac{1}{2}(1 - |\Phi|^2)\right)^2 + B(1 - |\Phi|^2) + |\mathrm{D}_1 \Phi|^2 + |\mathrm{D}_2\Phi|^2\right) \, \mathrm{d}^2 x.$$

We now dissect the terms one by one. We first use the definition of $B \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \mathrm{d}A$ and integration by parts to obtain

$$\int_{\mathbb{R}^2} (1 - |\Phi|^2) \, \mathrm{d}A = -\int_{\mathbb{R}^2} \mathrm{d}(1 - |\Phi|^2) \wedge A = 2\int_{\mathbb{R}^2} (\Phi, \mathrm{D}\Phi) \wedge A.$$

Alternatively, we can explicitly write

$$\begin{aligned}
\int_{\mathbb{R}^2} B(1 - |\Phi|^2) \, \mathrm{d}^2 x &= \int_{\mathbb{R}^2} (\partial_1 A_2 - \partial_2 A_1)(1 - |\Phi|^2) \, \mathrm{d}^2 x\\
&= \int_{\mathbb{R}^2} (A_2 \partial_1 |\Phi|^2 - A_1 \partial_2 |\Phi|^2) \, \mathrm{d}^2 x\\
&= 2\int_{\mathbb{R}^2} A_2(\Phi, \mathrm{D}_1 \Phi) - A_1(\Phi, \mathrm{D}_2 \Phi).
\end{aligned}$$

Ultimately, we want to obtain something that looks like $|\bar{\partial}_A \Phi|^2$. We can write this out as

$$(\mathrm{D}_1 \Phi + i \mathrm{D}_2 \Phi, \mathrm{D}_1 \Phi + i \mathrm{D}_2 \Phi) = |\mathrm{D}_1 \Phi|^2 + |\mathrm{D}_2 \Phi|^2 + 2(\mathrm{D}_1 \Phi, i \mathrm{D}_2 \Phi).$$

We note that $i\Phi$ and Φ are always orthogonal, and A_i is always a real coefficient. So we can write

$$\begin{aligned}
(\mathrm{D}_1 \Phi, i \mathrm{D}_2 \Phi) &= (\partial_1 \Phi - i A_1 \Phi, i \partial_2 \Phi + A_2 \Phi)\\
&= (\partial_1 \Phi, i \partial_2 \Phi) + A_2(\Phi, \partial_1 \Phi) - A_1(\Phi, \partial_2 \Phi).
\end{aligned}$$

We now use again the fact that $(\Phi, i\Phi) = 0$ to replace the usual derivatives with the covariant derivatives. So we have

$$(\mathrm{D}_1\Phi, i\mathrm{D}_2\Phi) = (\partial_1\Phi, i\partial_2\Phi) + A_2(\Phi, \mathrm{D}_1\Phi) - A_1(\Phi, \mathrm{D}_2\Phi).$$

This tells us we have

$$\int \left(B(1-|\Phi|^2) + |\mathrm{D}_1\Phi|^2 + |\mathrm{D}_2\Phi|^2\right)\,\mathrm{d}^2x = \int \left(4|\bar{\partial}_A\Phi|^2 + 2(\partial_1\Phi, i\partial_2\Phi)\right)\,\mathrm{d}^2x.$$

It then remains to show that $(\partial_1\Phi, i\partial_2\Phi) = j^0(\Phi)$. But we just write

$$\begin{aligned}(\partial_1\Phi_1 + i\partial_1\Phi_2, -\partial_2\Phi_2 + i\partial_2\Phi_1) &= -(\partial_1\Phi_1, \partial_2\Phi_2) + (\partial_1\Phi_2, \partial_2\Phi_1)\\ &= -j^0(\Phi)\\ &= -\det\begin{pmatrix}\partial_1\Phi_1 & \partial_2\Phi_1\\ \partial_1\Phi_2 & \partial_2\Phi_2\end{pmatrix}\end{aligned}$$

Then we are done. □

Corollary. For any (A, Φ) with winding number N, we always have $V_1(A, \Phi) \geq \pi N$, and those (A, Φ) that achieve this bound are exactly those that satisfy

$$\bar{\partial}_A\Phi = 0, \quad B = \frac{1}{2}(1-|\Phi|^2).$$

Theorem. In the above situation, the Bogomolny equation $B = \frac{1}{2}(1-|\Phi|^2)$ is equivalent to the scalar equation for u

$$-\Delta u + (e^u - 1) = -4\pi\sum_{j=1}^N \delta_{Z_j}.$$

This is known as *Taubes' equation*.

Proof. We have

$$\begin{aligned}B &= \partial_1A_2 - \partial_2A_1\\ &= -\frac{1}{2}\partial_1^2u - \frac{1}{2}\partial_2^2u + \frac{1}{2}(\partial_1\partial_2 - \partial_2\partial_1)\theta\\ &= -\frac{1}{2}\Delta u + \frac{1}{2}(\partial_1\partial_2 - \partial_2\partial_1)\theta.\end{aligned}$$

We might think the second term vanishes identically, but that is not true. Our θ has some singularities, and so that expression is not going to vanish at the singularities. The precise statement is that $(\partial_1\partial_2 - \partial_2\partial_1)\theta$ is a distribution supported at the points Z_j.

To figure out what it is, we have to integrate:

$$\begin{aligned}\int_{|z-Z_j|\leq\varepsilon}(\partial_1\partial_2 - \partial_2\partial_1)\theta\,\mathrm{d}^2x &= \int_{|z-Z_j|=\varepsilon}\partial_1\theta\,\mathrm{d}x^1 + \partial_2\theta\,\mathrm{d}x^2\\ &= \oint_{|z-Z_j|=\varepsilon}\mathrm{d}\theta = 4\pi n_j,\end{aligned}$$

where n_j is the multiplicity of the zero. Thus, we deduce that

$$(\partial_1\partial_2 - \partial_2\partial_1)\theta = 2\pi\sum\delta_{Z_j}.$$

But then we are done! □

2.5 Physics of vortices

Theorem (Derrick's scaling argument)**.** Consider a field theory in d-dimensions with energy functional

$$E[\Phi] = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\Phi|^2 + U(\Phi)\right) \mathrm{d}^d x = T + W,$$

with T the integral of the gradient term and W the integral of the term involving U.

– If $d = 1$, then any stationary point must satisfy

$$T = W.$$

– If $d = 2$, then all stationary points satisfy $W = 0$.

– If $d \geq 3$, then all stationary points have $T = W = 0$, i.e. Φ is constant.

Proof. Suppose Φ were such a stationary point. Then for any variation Φ_λ of Φ such that $\Phi = \Phi_1$, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=1} E[\Phi_\lambda] = 0.$$

Consider the particular variation given by

$$\Phi_\lambda(\mathbf{x}) = \Phi(\lambda\mathbf{x}).$$

Then we have

$$W[\Phi_\lambda] = \int_{\mathbb{R}^d} U(\Phi_\lambda(\mathbf{x}))\ \mathrm{d}^d x = \lambda^{-d} \int_{\mathbb{R}^d} U(\Phi(\lambda\mathbf{x}))\ \mathrm{d}^d(\lambda x) = \lambda^{-d} W[\Phi].$$

On the other hand, since T contains two derivatives, scaling the derivatives as well gives us

$$T[\Phi_\lambda] = \lambda^{2-d} T[\Phi].$$

Thus, we find

$$E[\Phi_\lambda] = \lambda^{2-d} T[\Phi] + \lambda^{-d} W[\Phi].$$

Differentiating and setting $\lambda = 1$, we see that we need

$$(2 - d)T[\Phi] - dW[\Phi] = 0.$$

Then the results in different dimensions follow. □

2.6 Jackiw–Pi vortices

3 Skyrmions

3.1 Skyrme field and its topology

Theorem (Derrick's theorem)**.** We have $E_2 = E_4$ for any finite-energy static solution for $m = 0$ Skyrmions.

Proof. Suppose $U(\mathbf{x})$ minimizes $E = E_2 + E_4$. We rescale this solution, and define

$$\tilde{U}(\mathbf{x}) = U(\lambda \mathbf{x}).$$

Since U is a solution, the energy is stationary with respect to λ at $\lambda = 1$.

We can take the derivative of this and obtain

$$\partial_i \tilde{U}(\mathbf{x}) = \lambda \tilde{U}(\lambda \mathbf{x}).$$

Therefore we find

$$\tilde{R}_i(\mathbf{x}) = \lambda R_i(\lambda \mathbf{x}),$$

and therefore

$$\begin{aligned}
\tilde{E}_2 &= -\frac{1}{2} \int \mathrm{Tr}(\tilde{R}_i \tilde{R}_i)\; \mathrm{d}^3 x \\
&= -\frac{1}{2} \lambda^2 \int \mathrm{Tr}(R_i R_i)(\lambda \mathbf{x})\; \mathrm{d}^3 x \\
&= -\frac{1}{2} \frac{1}{\lambda} \int \mathrm{Tr}(R_i R_i)(\lambda \mathbf{x})\; \mathrm{d}^3 (\lambda x) \\
&= \frac{1}{\lambda} E_2.
\end{aligned}$$

Similarly,

$$\tilde{E}_4 = \lambda E_4.$$

So we find that

$$\tilde{E} = \frac{1}{\lambda} E_2 + \lambda E_4.$$

But this function has to have a minimum at $\lambda = 1$. So the derivative with respect to λ must vanish at 1, requiring

$$0 = \frac{\mathrm{d}\tilde{E}}{\mathrm{d}\lambda} = -\frac{1}{\lambda^2} E_2 + E_4 = 0$$

at $\lambda = 1$. Thus we must have $E_4 = E_2$. $\square$

3.2 Skyrmion solutions

3.3 Other Skyrmion structures

3.4 Asymptotic field and forces for $B = 1$ hedgehogs

3.5 Fermionic quantization of the $B = 1$ hedgehog

3.6 Rigid body quantization