

Part III — Classical and Quantum Solitons

Theorems

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Solitons are solutions of classical field equations with particle-like properties. They are localised in space, have finite energy and are stable against decay into radiation. The stability usually has a topological explanation. After quantisation, they give rise to new particle states in the underlying quantum field theory that are not seen in perturbation theory. We will focus mainly on kink solitons in one space dimension, on gauge theory vortices in two dimensions, and on Skyrmions in three dimensions.

Pre-requisites

This course assumes you have taken Quantum Field Theory and Symmetries, Fields and Particles. The small amount of topology that is needed will be developed during the course.

Reference

N. Manton and P. Sutcliffe, *Topological Solitons*, CUP, 2004

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0 Introduction

1 ϕ^4 kinks

1.1 Kink solutions

1.2 Dynamic kink

1.3 Soliton interactions

1.4 Quantization of kink motion

1.5 Sine-Gordon kinks

2 Vortices

2.1 Topological background

2.2 Global $U(1)$ Ginzburg–Landau vortices

Lemma. Assume Φ is a smooth solution of the ungauged Ginzburg–Landau equation in some domain. Then at any interior maximum point x_* of $|\Phi|$, we have $|\Phi(x_*)| \leq 1$.

2.3 Abelian Higgs/Gauged Ginzburg–Landau vortices

Proposition. If f is a smooth real-valued function, and Φ and Ψ are smooth complex scalar fields, then

$$\begin{aligned} D(f\Phi) &= (df)\Phi + fD\Phi, \\ d(\Phi, \Psi) &= (D\Phi, \Psi) + (\Phi, D\Psi). \end{aligned}$$

(Here (\cdot) is the real inner product defined above.) In coordinates, these take the form

$$\begin{aligned} D_j(f\Phi) &= (\partial_j f)\Phi + fD_j\Phi \\ \partial_j(\Phi, \Psi) &= (D_j\Phi, \Psi) + (\Phi, D_j\Psi). \end{aligned}$$

Proposition. If Φ is a smooth scalar field, then

$$(D_1D_2 - D_2D_1)\Phi = -iB\Phi.$$

Proposition.

$$DD\Phi = -iF\Phi.$$

Lemma. Assume Φ is a smooth solution of the gauged Ginzburg–Landau equation in some domain. Then at any interior maximum point x_* of $|\Phi|$, we have $|\Phi(x_*)| \leq 1$.

2.4 Bogomolny/self-dual vortices and Taubes' theorem

Theorem (Taubes' theorem). For $\lambda = 1$, the space of (gauge equivalence classes of) solutions of the Euler–Lagrange equations $\delta V_1 = 0$ with winding number N is $\mathcal{M} \cong \mathbb{C}^N$.

To be precise, given $N \in \mathbb{N}$ and an unordered set of points $\{Z_1, \dots, Z_N\}$, there exists a smooth solution $A(x; Z_1, \dots, Z_N)$ and $\Phi(x; Z_1, \dots, Z_N)$ which solves the Euler–Lagrange equations $\delta V_1 = 0$, and also the so-called *Bogomolny equations*

$$D_1\Phi + iD_2\Phi = 0, \quad B = \frac{1}{2}(1 - |\Phi|^2).$$

Moreover, Φ has exactly N zeroes Z_1, \dots, Z_N counted with multiplicity, where (using the complex coordinates $z = x_1 + ix_2$)

$$\Phi(x; Z_1, \dots, Z_N) \sim c_j(z - Z_j)^{n_j}$$

as $z \rightarrow Z_j$, where $n_j = |\{k : Z_k = Z_j\}|$ is the multiplicity and c_j is a nonzero complex number.

This is the unique such solution up to gauge equivalence. Furthermore,

$$V_1(A(\cdot, Z_1, \dots, Z_N), \Phi(\cdot; Z_1, \dots, Z_N)) = \pi N \quad (*)$$

and

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} B \, d^2x = N = \text{winding number.}$$

Finally, this gives all finite energy solutions of the gauged Ginzburg–Landau equations.

Lemma. We have

$$V_1(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^2} \left(\left(B - \frac{1}{2}(1 - |\Phi|^2) \right)^2 + 4|\bar{\partial}_A \Phi|^2 \right) d^2x + \pi N,$$

where

$$\bar{\partial}_A \Phi = \frac{1}{2}(D_1 \Phi + iD_2 \Phi).$$

Corollary. For any (A, Φ) with winding number N , we always have $V_1(A, \Phi) \geq \pi N$, and those (A, Φ) that achieve this bound are exactly those that satisfy

$$\bar{\partial}_A \Phi = 0, \quad B = \frac{1}{2}(1 - |\Phi|^2).$$

Theorem. In the above situation, the Bogomolny equation $B = \frac{1}{2}(1 - |\Phi|^2)$ is equivalent to the scalar equation for u

$$-\Delta u + (e^u - 1) = -4\pi \sum_{j=1}^N \delta_{Z_j}.$$

This is known as *Taubes' equation*.

2.5 Physics of vortices

Theorem (Derrick's scaling argument). Consider a field theory in d -dimensions with energy functional

$$E[\Phi] = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla \Phi|^2 + U(\Phi) \right) d^d x = T + W,$$

with T the integral of the gradient term and W the integral of the term involving U .

- If $d = 1$, then any stationary point must satisfy

$$T = W.$$

- If $d = 2$, then all stationary points satisfy $W = 0$.
- If $d \geq 3$, then all stationary points have $T = W = 0$, i.e. Φ is constant.

2.6 Jackiw–Pi vortices

3 Skyrmions

3.1 Skyrme field and its topology

Theorem (Derrick's theorem). We have $E_2 = E_4$ for any finite-energy static solution for $m = 0$ Skyrmions.

3.2 Skyrmion solutions

3.3 Other Skyrmion structures

3.4 Asymptotic field and forces for $B = 1$ hedgehogs

3.5 Fermionic quantization of the $B = 1$ hedgehog

3.6 Rigid body quantization