# Part IB — Quantum Mechanics

## Theorems

Based on lectures by J. M. Evans

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

**Physical background**
Photoelectric effect. Electrons in atoms and line spectra. Particle diffraction. [1]

**Schrödinger equation and solutions**
De Broglie waves. Schrödinger equation. Superposition principle. Probability interpretation, density and current. [2]

Stationary states. Free particle, Gaussian wave packet. Motion in 1-dimensional potentials, parity. Potential step, square well and barrier. Harmonic oscillator. [4]

**Observables and expectation values**
Position and momentum operators and expectation values. Canonical commutation relations. Uncertainty principle. [2]

Observables and Hermitian operators. Eigenvalues and eigenfunctions. Formula for expectation value. [2]

**Hydrogen atom**
Spherically symmetric wave functions for spherical well and hydrogen atom.

Orbital angular momentum operators. General solution of hydrogen atom. [5]

# Contents

# 0 Introduction

## 0.1 Light quanta

## 0.2 Bohr model of the atom

## 0.3 Matter waves

# 1 Wavefunctions and the Schrödinger equation

## 1.1 Particle state and probability

## 1.2 Operators

## 1.3 Time evolution of wavefunctions

**Proposition.** The probability density

$$P(x,t) = |\Psi(x,t)|^2$$

obeys a conservation equation

$$\frac{\partial P}{\partial t} = -\frac{\partial j}{\partial x},$$

where

$$j(x,t) = -\frac{i\hbar}{2m}\left(\Psi^* \frac{\mathrm{d}\Psi}{\mathrm{d}x} - \frac{\mathrm{d}\Psi^*}{\mathrm{d}x}\Psi\right)$$

is the *probability current*.

# 2 Some examples in one dimension

## 2.1 Introduction

## 2.2 Infinite well — particle in a box

## 2.3 Parity

## 2.4 Potential well

## 2.5 The harmonic oscillator

# 3 Expectation and uncertainty

## 3.1 Inner products and expectation values

**Proposition.** The operators $\hat{x}$, $\hat{p}$ and $H$ are all Hermitian (for real potentials).

**Proposition** (Cauchy-Schwarz inequality)**.** If $\psi$ and $\phi$ are any normalizable states, then

$$\|\psi\|\|\phi\| \geq |(\psi, \phi)|.$$

## 3.2 Ehrenfest's theorem

**Theorem** (Ehrenfest's theorem)**.**

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{x}\rangle_\Psi &= \frac{1}{m}\langle \hat{p}\rangle_\Psi\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{p}\rangle_\Psi &= -\langle V'(\hat{x})\rangle_\Psi.\end{aligned}$$

## 3.3 Heisenberg's uncertainty principle

**Theorem** (Heisenberg's uncertainty principle)**.** If $\psi$ is any normalized state (at any fixed time), then

$$(\Delta x)_\psi (\Delta p)_\psi \geq \frac{\hbar}{2}.$$

# 4 More results in one dimensions

## 4.1 Gaussian wavepackets

## 4.2 Scattering

## 4.3 Potential step

## 4.4 Potential barrier

## 4.5 General features of stationary states

# 5 Axioms for quantum mechanics

## 5.1 States and observables

## 5.2 Measurements

**Proposition.** Let $Q$ be Hermitian (an observable), i.e. $Q^\dagger = Q$. Then

(i) Eigenvalues of $Q$ are real.

(ii) Eigenstates of $Q$ with different eigenvalues are orthogonal (with respect to the complex inner product).

(iii) Any state can be written as a (possibly infinite) linear combination of eigenstates of $Q$, i.e. eigenstates of $Q$ provide a basis for $V$. Alternatively, the set of eigenstates is *complete*.

**Proposition.**

(i) The expectation value of $Q$ in state $\psi$ is

$$\langle Q\rangle_\psi = (\psi, Q\psi) = \sum \lambda_n P_n,$$

with notation as in the previous part.

(ii) The uncertainty $(\delta Q)_\psi$ is given by

$$(\Delta Q)^2_\psi = \langle (Q - \langle Q\rangle_\psi)^2\rangle_\psi = \langle Q^2\rangle_\psi - \langle Q\rangle_\psi^2 = \sum_n (\lambda_n - \langle Q\rangle_\psi)^2 P_n.$$

## 5.3 Evolution in time

**Theorem** (Ehrenfest's theorem)**.** If $Q$ is any operator with no explicit time dependence, then

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\langle Q\rangle_\Psi = \langle [Q, H]\rangle_\Psi,$$

where

$$[Q, H] = QH - HQ$$

is the commutator.

## 5.4 Discrete and continuous spectra

## 5.5 Degeneracy and simultaneous measurements

# 6 Quantum mechanics in three dimensions

## 6.1 Introduction

## 6.2 Separable eigenstate solutions

## 6.3 Angular momentum

**Proposition.**

(i) $[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$.

(ii) $[\mathbf{L}^2, L_i] = 0$.

(iii) $[L_i, \hat{x}_j] = i\hbar\varepsilon_{ijk}\hat{x}_k$ and $[L_i, \hat{p}_j] = i\hbar\varepsilon_{ijk}\hat{p}_k$

## 6.4 Joint eigenstates for a spherically symmetric potential

# 7 The hydrogen atom

## 7.1 Introduction

## 7.2 General solution

## 7.3 Comments