Example Sheet 1

Values of some physical constants are given on the supplementary sheet

1. When a sample of potassium is illuminated with light of wavelength 3×10^{-7}m, electrons are emitted with kinetic energy 2.1 eV. When the same sample is illuminated with light of wavelength 5×10^{-7}m it emits electrons with kinetic energy 0.5 eV. Use Einstein's explanation of this *photoelectric effect* to obtain a value for Planck's constant, and find the minimum energy W needed to free an electron from the surface of potassium.

2. The light from a faint star has energy flux $10^{-10}\,\mathrm{Jm^{-2}s^{-1}}$. If the wavelength of the light is 5×10^{-7}m, estimate the number of photons from this star that enter a human eye in one second.

3. Consider the Bohr model of the Hydrogen atom, taking the electron to be a non-relativistic point particle of mass m travelling with speed v in a circular orbit of radius r around a point-like proton. The inward acceleration v^2/r must be provided by the Coulomb attraction $e^2/4\pi\epsilon_0 r^2$, and the angular momentum is assumed to be quantised: $L = mvr = n\hbar$, with $n = 1, 2, 3, \ldots$. Show that the possible energy levels for the electron are

$$E = -\frac{1}{2}mc^2\alpha^2\frac{1}{n^2}$$

where α is the fine structure constant.

(i) Is the result for v consistent with the assumption that the motion of the electron is non-relativistic?

(ii) Suppose that the electron makes a transition from one energy level to another, emitting a photon in the process. What is the smallest possible wavelength for the emitted photon, and how does this compare to the Bohr radius r_1 (corresponding to $n = 1$)?

4. A muon is a particle with the same charge as an electron, but with a mass about 207 times larger. A muon can be captured by a proton to form an atom of muonic Hydrogen. How does the radius of the $n = 1$ state orbit compare to that of ordinary Hydrogen?

5. (*This example requires some Special Relativity - do it if time allows*) A photon of wavelength λ scatters off an electron which is initially stationary in the laboratory frame. If λ' is the wavelength of the photon after the collision, and θ is the angle through which it is deflected, show that conservation of four-momentum implies

$$\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)\ .$$

What is the value of $h/m_e c$? (the *Compton wavelength* of the electron)

6. The time-independent Schrödinger Equation for a one-dimensional harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}Kx^2\psi \;=\; E\psi\ .$$

where m is the mass and the constant K gives the strength of the restoring force. Verify that there are energy eigenfunctions of the form

$$\psi_0(x) \;=\; C_0\, e^{-x^2/2\alpha}\ , \qquad \psi_1(x) \;=\; C_1\, x e^{-x^2/2\alpha}$$

for a certain value of α, to be determined, and find the corresponding energy eigenvalues E_0 and E_1. Sketch ψ_0, ψ_1, and $V(x)$. (C_0 and C_1 are normalisation constants that you need not determine.)

7. A particle of mass m moves freely in one dimension ($V = 0$). Consider the wavefunction

$$\Psi(x,t) \;=\; (\alpha/\pi)^{1/4}\,\gamma(t)^{-1/2}\,\exp(\,-x^2/2\gamma(t)\,)\,,$$

where $\gamma(t)$ is complex valued, and α is a real positive constant. By substituting this into the time-dependent Schrödinger equation, find a necessary and sufficient condition on $\gamma(t)$ for $\Psi(x,t)$ to be a solution. Hence determine $\gamma(t)$ if $\gamma(0) = \alpha$. Check that $\Psi(x,t)$ is then correctly normalised for all t.

Write down an expression for the probability of finding the particle in an interval $a \le x \le b$ and show that this vanishes in the limit $t \to \infty$ (with a and b held fixed).

8. Write down the time-independent Schrödinger equation for the wavefunction of a particle moving in a potential $V = -U\delta(x)$, where U is a positive constant and $\delta(x)$ is the Dirac delta function. Integrate the equation over the interval $-\epsilon < x < \epsilon$, for a positive constant ϵ, and hence deduce that there is a discontinuity at $x = 0$ in the derivative of $\psi(x)$:

$$\lim_{\epsilon\to 0}\,[\,\psi'(\epsilon) - \psi'(-\epsilon)\,] \;=\; -\frac{2mU}{\hbar^2}\psi(0)\ .$$

By using this condition to relate appropriate solutions for $x > 0$ and $x < 0$, find the unique normalisable eigenstate of the Hamiltonian, and determine its energy eigenvalue.

9. Consider a square well potential with $V(x) = -U$ for $|x| < a$ and $V(x) = 0$ otherwise (U is a positive constant). Show that there are no bound states (normalisable energy eigenfunctions) which satisfy $\psi(-x) = -\psi(x)$ (i.e. which have odd parity) if $a^2U < (\pi\hbar)^2/8m$.

10. Sketch the potential

$$V(x) = -\frac{\hbar^2}{m}\mathrm{sech}^2 x\ .$$

Show that the time-independent Schrödinger equation for a particle in this potential can be written

$$A^\dagger A\,\psi \;=\; (\mathcal{E}+1)\psi$$

where $\mathcal{E} = 2mE/\hbar^2$ and

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + \tanh x\ , \quad A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + \tanh x\ .$$

Show, by integrating by parts, that for any normalised wavefunction ψ,

$$\int_{-\infty}^{\infty} \psi^* A^\dagger A\psi\,\mathrm{d}x \;=\; \int_{-\infty}^{\infty} (A\psi)^*(A\psi)\,\mathrm{d}x$$

and deduce that the eigenvalues of $A^\dagger A$ are non-negative. Hence show that the ground state (with lowest energy) has $\mathcal{E} \ge -1$. Show that a wavefunction $\psi_0(x)$ is an energy eigenstate with $\mathcal{E} = -1$ iff

$$\frac{\mathrm{d}\psi_0}{\mathrm{d}x} + \tanh x\,\psi_0 \;=\; 0\ .$$

Find and sketch $\psi_0(x)$.

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Mathematical Tripos Part IB **Michaelmas Term 2015**
Quantum Mechanics **Dr. J.M. Evans**

Example Sheet 2

1. A particle of mass m is confined to a one-dimensional box $0 \leq x \leq a$ (the potential $V(x)$ is zero inside the box, and infinite outside). Find the normalised energy eigenstates $\psi_n(x)$ and the corresponding energy eigenvalues E_n. Show that the expectation value and the uncertainty for a measurement of $\hat{x}$ in the state ψ_n are

$$\langle \hat{x} \rangle_n = \frac{a}{2} \qquad \text{and} \qquad (\Delta x)_n^2 = \frac{a^2}{12}\left(1 - \frac{6}{\pi^2 n^2}\right) .$$

Does the limit $n \to \infty$ agree with what you would expect for a classical particle in this potential?

2. Write down the Hamiltonian for a harmonic oscillator of mass m and frequency ω. Express $\langle H \rangle$ in terms of $\langle \hat{x} \rangle$, $\langle \hat{p} \rangle$, Δx and Δp (all defined for some normalised state ψ). Use the Uncertainty Relation to deduce that $E \geq \frac{1}{2}\hbar\omega$ for any energy eigenvalue E.

3. Let $\psi_1(x)$ and $\psi_2(x)$ be normalisable energy eigenfunctions for a particle of mass m in one dimension moving in a potential $V(x)$. Suppose that ψ_1 and ψ_2 have the same energy eigenvalue E. By considering $\psi_1 \psi'_2 - \psi_2 \psi'_1$, show that these wavefunctions must be proportional to one another. What does this mean, physically? Can the wavefunction for a normalisable energy eigenstate (bound state) in one dimension always be chosen to be real?

4. Consider a particle of mass m in one dimension with potential energy $V(x)$. Show that if the particle is in a stationary state, then the probability current j is independent of x.

Now suppose that a stationary state $\psi(x)$ corresponds to scattering by the potential, and that $V(x) \to 0$ as $x \to \pm\infty$. Given the asymptotic behaviour

$$\psi(x) \sim e^{ikx} + Ae^{-ikx} \quad (x \to -\infty) \qquad \text{and} \qquad \psi(x) \sim Be^{ikx} \quad (x \to +\infty)$$

show that $|A|^2 + |B|^2 = 1$. How should you interpret this?

5. A particle is incident on a potential barrier of width a and height U. Assuming that $U = 2E$, where $E = \hbar^2 k^2 / 2m$ is the kinetic energy of the incident particle, find the transmission probability.
[*Work through the algebra, which simplifies in this particular case, rather than quoting the result given in lectures.*]

6. Consider the time-independent Schrödinger equation with potential $V(x) = -(\hbar^2/m)\,\mathrm{sech}^2 x$. Show that

$$\psi(x) = e^{ikx}(\tanh x - ik)$$

is a solution for any real k, and find the energy. Show that ψ is the wavefunction of a scattering state for which the reflection amplitude vanishes. Find the transmission amplitude $A_{\rm tr}(k)$, and verify that the transmission probability is 1.

Setting $k = i\kappa$, show that for a certain real, positive value of κ, the wavefunction $\psi(x)$ becomes a bound state solution. What is its energy? How does the function $A_{\rm tr}(k)$ behave at this value of k?

7. A particle of mass m is in a one-dimensional infinite square well (a potential box) with $V = 0$ for $0 < x < a$ and $V = \infty$ otherwise. The normalised wavefunction for the particle at time $t = 0$ is

$$\Psi(x, 0) = Cx(a - x) .$$

(i) Determine the real constant C.

7. (ii) By expanding $\Psi(x,0)$ as a linear combination of energy eigenfunctions (as found in a previous example), obtain an expression for $\Psi(x,t)$, the wavefunction at time t.

(iii) A measurement of the energy is made at time $t > 0$. Show that the probability that this yields the result $E_n = \hbar^2\pi^2n^2/2ma^2$ is $960/\pi^6n^6$ if n is odd, and zero if n is even. Why should this be expected for n even? Which value of the energy is most likely, and why is its probability so close to unity?

8. The energy levels of the harmonic oscillator are $E_n = (n+\frac{1}{2})\hbar\omega$ for $n = 0, 1, 2, \ldots$. The corresponding stationary state wavefunctions have the form

$$\psi_n(x) \;=\; h_n(y)e^{-y^2/2} \qquad \text{where} \qquad y = (m\omega/\hbar)^{1/2}x$$

and h_n is a polynomial of degree n, with $h_n(-y) = (-1)^n h_n(y)$. Use the orthogonality relation

$$(\psi_m, \psi_n) = \delta_{mn}$$

to determine ψ_2, up to an overall constant.

Suppose that the initial wavefunction for the oscillator $\Psi(x,0)$ has odd parity. By considering this as a linear combination of energy eigenfunctions, show that $|\Psi(x,t)|^2$ is periodic in t with period π/ω.

9. A quantum system has Hamiltonian H with normalised eigenstates ψ_n and corresponding energies E_n $(n = 1, 2, 3, \ldots)$. A linear operator Q is defined by its action on these states:

$$Q\psi_1 = \psi_2 \ , \qquad Q\psi_2 = \psi_1 \ , \qquad Q\psi_n = 0 \quad n > 2 \ .$$

Show that Q has eigenvalues ± 1 (in addition to zero) and find the corresponding normalised eigenstates $\chi_\pm$, in terms of energy eigenstates. Calculate $\langle H \rangle$ in each of the states $\chi_\pm$.

A measurement of Q is made at time zero, and the result $+1$ is obtained. The system is then left undisturbed for a time t, at which instant another measurement of Q is made. What is the probability that the result will again be $+1$? Show that the probability is zero if the measurement is made when a time $T = \pi\hbar/(E_2{-}E_1)$ has elapsed (assume $E_2{-}E_1 > 0$).

10. In the previous example, suppose that an experimenter makes n successive measurements of Q at regular time intervals T/n. If the result $+1$ is obtained for one measurement, show that the probability that the next measurement also gives $+1$ can be written as $|A_n|^2$, with

$$A_n = 1 \;-\; \frac{iT(E_1{+}E_2)}{2\hbar n} \;+\; \mathcal{O}\Big(\frac{1}{n^2}\Big)$$

The probability that all n measurements give the result $+1$ is therefore $P_n = (|A_n|^2)^n$. Show that

$$\lim_{n\to\infty} P_n = 1 \ .$$

Interpreting $\chi_\pm$ as the 'not-boiling' and 'boiling' states of a two-state 'quantum kettle', this shows that a watched quantum kettle never boils (also called the Quantum Zeno Paradox).

11. The Hamiltonian for a particle in one dimension is $H = T + V$ where $T = \hat{p}^2/2m$ is the kinetic energy and $V(\hat{x})$ is some potential. Show that the expectation value $\langle T \rangle$ is positive in any (normalized) state ψ. By considering $\langle H \rangle$, show that the lowest bound state (assuming there is one) has energy above the minimum value of $V(x)$.

Now consider expectation values in an eigenstate ψ of H with energy E. Show that

$$\langle \, [H, A] \, \rangle \;=\; 0$$

for any operator A. By choosing $A = \hat{x}$, deduce that $\langle \hat{p} \rangle = 0$.

For $V(\hat{x}) = k\hat{x}^n$ (with k and n constants) take $A = \hat{x}\hat{p}$ to derive the *Virial Theorem*

$$2\langle T \rangle \;=\; n\langle V \rangle \qquad \text{and deduce that} \qquad \langle T \rangle = \frac{n}{n+2}E \, .$$

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Example Sheet 3

1. A particle moving in three dimensions is confined within a box $0 < x < a$, $0 < y < b$, $0 < z < c$. (The potential is zero inside and infinite outside.) By considering stationary state wavefunctions of the form $\psi(x, y, z) = X(x)Y(y)Z(z)$, show that the allowed energy levels are

$$\frac{\hbar^2\pi^2}{2m}\Big(\frac{n_1^2}{a^2}+\frac{n_2^2}{b^2}+\frac{n_3^2}{c^2}\Big) \qquad \text{for integers } n_i > 0\ .$$

What is the degeneracy of the first excited energy level when $a = b = c$?

2. The isotropic 3-dimensional harmonic oscillator has potential $V(x_1, x_2, x_3) = \frac{1}{2}m\omega^2(x_1^2+x_2^2+x_3^2)$. Find energy eigenstates of separable form in Cartesian coordinates, and hence show that the energy levels are

$$E = (\, n_1 + n_2 + n_3 + \tfrac{3}{2}\,)\hbar\omega$$

where n_1, n_2, n_3 are non-negative integers. (Assume results for the oscillator in one dimension.)

How many linearly independent states have energy $E = \big(N + \frac{3}{2}\big)\,\hbar\omega$? Show that the ground state is spherically symmetric and find a state with $N = 2$ that is also spherically symmetric.

3. (i) Verify that $A\exp(i\mathbf{k}\cdot\mathbf{x})$ is a simultaneous eigenstate of the components of momentum $\hat{\mathbf{p}}$. Hence show that it is also an energy eigenstate for a free particle (vanishing potential).

(ii) Let $\Psi(\mathbf{x}, t)$ be any solution of the time-dependent Schrödinger Equation with zero potential. Show that

$$\Psi(\mathbf{x}-\mathbf{u}t, t)\exp(i\mathbf{k}\cdot\mathbf{x})\exp(-i\hbar\mathbf{k}^2 t/2m)$$

is also a solution, provided $\hbar\mathbf{k} = m\mathbf{u}$ (where m is the mass of the particle).

4. Suppose Q is an observable that does not depend explicitly on time. Show that

$$i\hbar\frac{d}{dt}\langle Q\rangle_\Psi = \langle\, [Q, H]\,\rangle_\Psi$$

where $\Psi(t)$ obeys the Schrödinger Equation. Apply this to the position and momentum of a particle in three dimensions, with Hamiltonian

$$H = \frac{1}{2m}\hat{\mathbf{p}}^2 + V(\hat{\mathbf{x}})\ ,$$

by calculating the commutator of H with each component of $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$. Compare the results with the classical equations of motion.

5. The time-independent Schrödinger Equation for an electron in a Hydrogen atom is

$$-\frac{\hbar^2}{2m}\nabla^2\psi \;-\; \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\,\psi \;=\; E\psi$$

Show that there is a spherically-symmetric energy eigenstate of the form

$$\psi(r) = Ce^{-r/a}$$

for a certain value of the constant a, and find the corresponding energy eigenvalue E.

Compute the value of C required to normalise the wavefunction. What is the expectation value of the distance of the electron from the proton (which is assumed to be stationary at the origin) and how does this compare to the Bohr radius? [*Recall that* $\nabla^2 f = f'' + (2/r)f'$ for $f(r)$.]

6. As a model for the deuterium nucleus (a bound state of a proton and a neutron) consider a particle in the 3-dimensional square-well potential

$$V(r) = \begin{cases} -U & r < a \\ 0 & r > a \end{cases}$$

with $U > 0$. Consider the radial Schrödinger equation for the special case of a spherically symmetric wavefunction. Is there always a spherically symmetric bound state solution?

7. Let A and B be hermitian operators. Show that $i[A, B]$ is hermitian.

Let Ψ be any normalised state. Setting $\Phi = (A + i\lambda B)\Psi$ and considering (Φ, Φ) as a quadratic in the real variable λ, show that

$$\langle A^2 \rangle \langle B^2 \rangle \geq \tfrac{1}{4} \, | \, \langle \, i[A, B] \, \rangle \, |^2$$

(with all expectation values taken in the state Ψ). Hence derive the generalised Uncertainty Principle:

$$\Delta A \, \Delta B \geq \tfrac{1}{2} \, | \, \langle \, [A, B] \, \rangle \, | \; .$$

8. Let $\phi(r)$ be any spherically symmetric wavefunction. Show, using Cartesian coordinates, that $L_3 \, \phi = 0$. Show that $\phi(r)$ is also an eigenstate of L^2. [*Recall that* $\partial r / \partial x_i = x_i / r$.]

Now calculate the results obtained by applying L_3 and L_3^2 to each of the wavefunctions

$$\psi_i(\mathbf{x}) = x_i \, \phi(r) \qquad \text{with} \qquad i = 1, 2, 3 \, .$$

From your answers, deduce that each $\psi_i(\mathbf{x})$ is an eigenfunction of L^2 with eigenvalue $2\hbar^2$. Find linear combinations of the wavefunctions $\psi_i(\mathbf{x})$ that are also eigenfunctions of L_3; what are the eigenvalues?

9. Use the commutation relations for orbital angular momentum $[\, L_1, \, L_2 \,] = i\hbar L_3$ (and cyclic permutations) to show that $[\, L_3, \, L^2 \,] = 0$.

Prove that $\langle \, [\, L_3, \, A \,] \, \rangle = 0$ when the expectation value is taken in any eigenstate of L_3, for any operator A. Hence, by evaluating $[\, L_3, \, L_1 L_2 \,]$, deduce that $\langle \, L_1^2 \, \rangle = \langle \, L_2^2 \, \rangle$ in any eigenstate of L_3. Now consider a joint eigenstate for which L_3 has eigenvalue $\hbar m$ and L^2 has eigenvalue $\hbar^2 \ell(\ell{+}1)$. Show that $\langle \, L_1^2 \, \rangle = \langle \, L_2^2 \, \rangle = \frac{1}{2}\hbar^2 (\, \ell(\ell{+}1) - m^2 \,)$ in this state.

10. Consider the $2{\times}2$ hermitian matrices defined by

$$S_1 = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad S_2 = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad S_3 = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Evaluate the commutators $[\, S_i, \, S_j \,]$ for all values of i and j (there are only three independent cases to consider) and calculate the matrix $S^2 = S_1^2 + S_2^2 + S_3^2$.

Write down simultaneous eigenvectors of S_3 and S^2 and hence show that their eigenvalues are $\pm\hbar s$ and $\hbar^2 s(s+1)$, respectively, for a certain positive number s.

11. Suppose that the Hamiltonian of a quantum system depends on a parameter that changes suddenly, at a certain time, by a finite amount. Show that the wavefunction must change continuously if the time-dependent Schrödinger equation is to be valid throughout the change.

In a *hydrogenic atom*, a single electron is bound to a nucleus of charge Ze, with Z a positive integer. The normalised ground state wavefunction has the form

$$\psi(r) = \frac{c}{\sqrt{a^3}} \, e^{-r/a} \; .$$

From your answer to question **5**, give the value of c and find the dependence of a on Z.

A hydrogenic atom is in its ground state when the nucleus emits an electron, suddenly changing its charge from Ze to $(Z+1)e$. Calculate the probability that a measurement of the energy of the atom after the emission will also find it to be in its ground state.

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