Part IB — Methods

Theorems with proof

Based on lectures by D. B. Skinner

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Self-adjoint ODEs
Periodic functions. Fourier series: definition and simple properties; Parseval's theorem. Equations of second order. Self-adjoint differential operators. The Sturm-Liouville equation; eigenfunctions and eigenvalues; reality of eigenvalues and orthogonality of eigenfunctions; eigenfunction expansions (Fourier series as prototype), approximation in mean square, statement of completeness. [5]

PDEs on bounded domains: separation of variables
Physical basis of Laplace's equation, the wave equation and the diffusion equation. General method of separation of variables in Cartesian, cylindrical and spherical coordinates. Legendre's equation: derivation, solutions including explicit forms of P_0, P_1 and P_2, orthogonality. Bessel's equation of integer order as an example of a self-adjoint eigenvalue problem with non-trivial weight.

Examples including potentials on rectangular and circular domains and on a spherical domain (axisymmetric case only), waves on a finite string and heat flow down a semi-infinite rod. [5]

Inhomogeneous ODEs: Green's functions
Properties of the Dirac delta function. Initial value problems and forced problems with two fixed end points; solution using Green's functions. Eigenfunction expansions of the delta function and Green's functions. [4]

Fourier transforms
Fourier transforms: definition and simple properties; inversion and convolution theorems. The discrete Fourier transform. Examples of application to linear systems. Relationship of transfer function to Green's function for initial value problems. [4]

PDEs on unbounded domains
Classification of PDEs in two independent variables. Well posedness. Solution by the method of characteristics. Green's functions for PDEs in 1, 2 and 3 independent variables; fundamental solutions of the wave equation, Laplace's equation and the diffusion equation. The method of images. Application to the forced wave equation, Poisson's equation and forced diffusion equation. Transient solutions of diffusion problems: the error function. [6]

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0 Introduction

1 Vector spaces

2 Fourier series

2.1 Fourier series

2.2 Convergence of Fourier series

2.3 Differentiability and Fourier series

Theorem (Parseval's theorem)**.**

$$(f, f) = \int_{-\pi}^{\pi} |f(\theta)|^2 \, \mathrm{d}\theta = 2\pi \sum_{n \in \mathbb{Z}} |\hat{f}_n|^2$$

Proof.

$$\begin{aligned}
(f, f) &= \int_{-\pi}^{\pi} |f(\theta)|^2 \, \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \left(\sum_{m \in \mathbb{Z}} \hat{f}_m^* e^{-im\theta} \right) \left(\sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta} \right) \, \mathrm{d}\theta \\
&= \sum_{m, n \in \mathbb{Z}} \hat{f}_m^* \hat{f}_n \int_{-\pi}^{\pi} e^{i(n-m)\theta} \, \mathrm{d}\theta \\
&= 2\pi \sum_{m, n \in \mathbb{Z}} \hat{f}_m^* \hat{f}_n \delta_{mn} \\
&= 2\pi \sum_{n \in \mathbb{Z}} |\hat{f}_n|^2
\end{aligned}$$

□

3 Sturm-Liouville Theory

3.1 Sturm-Liouville operators

Proposition. The eigenvalues of a Sturm-Liouville operator are real.

Proof. Suppose $\mathcal{L}y_i = \lambda_i w y_i$. Then

$$\lambda_i(y_i, y_i)_w = \lambda_i(y_i, wy_i) = (y_i, \mathcal{L}y_i) = (\mathcal{L}y_i, y_i) = (\lambda_i w y_i, y_i) = \lambda_i^*(y_i, y_i)_w.$$

Since $(y_i, y_i)_w \neq 0$, we have $\lambda_i = \lambda_i^*$. □

Proposition. Eigenfunctions with different eigenvalues (but same weight) are orthogonal.

Proof. Let $\mathcal{L}y_i = \lambda_i w y_i$ and $\mathcal{L}y_j = \lambda_j w y_j$. Then

$$\lambda_i(y_j, y_i)_w = (y_j, \mathcal{L}y_i) = (\mathcal{L}y_j, y_i) = \lambda_j(y_j, y_i)_w.$$

Since $\lambda_i \neq \lambda_j$, we must have $(y_j, y_i)_w = 0$. □

Theorem. On a *compact* domain, the eigenvalues $\lambda_1, \lambda_2, \cdots$ form a countably infinite sequence and are discrete.

Theorem. The eigenfunctions are complete: any function $f : [a, b] \to \mathbb{C}$ (obeying appropriate boundary conditions) can be expanded as

$$f(x) = \sum_n \hat{f}_n y_n(x),$$

where

$$\hat{f}_n = \int y_n^*(x) f(x) w(x)\;\mathrm{d}x.$$

Theorem (Parseval's theorem II)**.**

$$(f, f)_w = \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2.$$

Proof. We have

$$\begin{aligned}(f, f)_w &= \int_\Omega f^*(x) f(x) w(x)\;\mathrm{d}x\\ &= \sum_{n,m\in\mathbb{Z}} \int_\Omega \hat{f}_n^* y_n^*(x) \hat{f}_m y_m(x) w(x)\;\mathrm{d}x\\ &= \sum_{n,m\in\mathbb{Z}} \hat{f}_n^* \hat{f}_m (y_n, y_m)_w\\ &= \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2.\end{aligned}$$

□

3.2 Least squares approximation

4 Partial differential equations

4.1 Laplace's equation

Proposition. Let Ω be a compact domain, and $\partial\Omega$ be its boundary. Let $f : \partial\Omega \to \mathbb{R}$. Then there is a unique solution to $\nabla^2\phi = 0$ on Ω with $\phi|_{\partial\Omega} = f$.

Proof. Suppose ϕ_1 and ϕ_2 are both solutions such that $\phi_1|_{\partial\Omega} = \phi_2|_{\partial\Omega} = f$. Then let $\Phi = \phi_1 - \phi_2$. So $\Phi = 0$ on the boundary. So we have

$$0 = \int_\Omega \Phi\nabla^2\Phi \;\mathrm{d}^n x = -\int_\Omega (\nabla\Phi)\cdot(\nabla\Phi)\;\mathrm{d}x + \int_{\partial\Omega} \Phi\nabla\Phi\cdot\mathbf{n}\;\mathrm{d}^{n-1}x.$$

We note that the second term vanishes since $\Phi = 0$ on the boundary. So we have

$$0 = -\int_\Omega (\nabla\Phi)\cdot(\nabla\Phi)\;\mathrm{d}x.$$

However, we know that $(\nabla\Phi)\cdot(\nabla\Phi) \geq 0$ with equality iff $\nabla\Phi = 0$. Hence Φ is constant throughout Ω. Since at the boundary, $\Phi = 0$, we have $\Phi = 0$ throughout, i.e. $\phi_1 = \phi_2$. □

4.2 Laplace's equation in the unit disk in $\mathbb{R}^2$

4.3 Separation of variables

4.4 Laplace's equation in spherical polar coordinates

4.4.1 Laplace's equation in spherical polar coordinates

4.4.2 Legendre Polynomials

4.4.3 Solution to radial part

4.5 Multipole expansions for Laplace's equation

4.6 Laplace's equation in cylindrical coordinates

4.7 The heat equation

Proposition. Suppose $\phi : \Omega\times[0,\infty) \to \mathbb{R}$ satisfies the heat equation $\frac{\partial\phi}{\partial t} = \kappa\nabla^2\phi$, and obeys

- Initial conditions $\phi(\mathbf{x}, 0) = f(x)$ for all $x \in \Omega$
- Boundary condition $\phi(\mathbf{x}, t)|_{\partial\Omega} = g(\mathbf{x}, t)$ for all $t \in [0,\infty)$.

Then $\phi(\mathbf{x}, t)$ is unique.

Proof. Suppose ϕ_1 and ϕ_2 are both solutions. Then define $\Phi = \phi_1 - \phi_2$ and

$$E(t) = \frac{1}{2}\int_\Omega \Phi^2\;\mathrm{d}V.$$

Then we know that $E(t) \geq 0$. Since ϕ_1, ϕ_2 both obey the heat equation, so does Φ. Also, on the boundary and at $t = 0$, we know that $\Phi = 0$. We have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_\Omega \Phi \frac{\mathrm{d}\Phi}{\mathrm{d}t}\ \mathrm{d}V \\ &= \kappa \int_\Omega \Phi \nabla^2 \Phi\ \mathrm{d}V \\ &= \kappa \int_{\partial\Omega} \Phi \nabla \Phi \cdot \mathrm{d}\mathbf{S} - \kappa \int_\Omega (\nabla \Phi)^2\ \mathrm{d}V \\ &= -\kappa \int_\Omega (\nabla \Phi)^2\ \mathrm{d}V \\ &\leq 0.\end{aligned}$$

So we know that E decreases with time but is always non-negative. We also know that at time $t = 0$, $E = \Phi = 0$. So $E = 0$ always. So $\Phi = 0$ always. So $\phi_1 = \phi_2$. □

4.8 The wave equation

Proposition. Suppose $\phi : \Omega \times [0, \infty) \to \mathbb{R}$ obeys the wave equation $\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi$ inside $\Omega \times (0, \infty)$, and is fixed at the boundary. Then E is constant.

Proof. By definition, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\Omega \frac{\partial^2 \psi}{\partial t^2} \frac{\partial \psi}{\partial t} + c^2 \nabla \left(\frac{\partial \phi}{\partial t}\right) \cdot \nabla \phi\ \mathrm{d}V.$$

We integrate by parts in the second term to obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\Omega \frac{\mathrm{d}\phi}{\mathrm{d}t} \left(\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi\right)\ \mathrm{d}V + c^2 \int_{\partial\Omega} \frac{\partial \phi}{\partial t} \nabla \phi \cdot \mathrm{d}S.$$

Since $\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi$ by the wave equation, and ϕ is constant on $\partial\Omega$, we know that

$$\frac{\mathrm{d}E_\phi}{\mathrm{d}t} = 0.$$

□

Proposition. Suppose $\phi : \Omega \times [0, \infty) \to \mathbb{R}$ obeys the wave equation $\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi$ inside $\Omega \times (0, \infty)$, and obeys, for some f, g, h,

(i) $\phi(x, 0) = f(x)$;

(ii) $\frac{\partial \phi}{\partial t}(x, 0) = g(x)$; and

(iii) $\phi|_{\partial\Omega \times [0, \infty)} = h(x)$.

Then ϕ is unique.

Proof. Suppose ϕ_1 and ϕ_2 are two such solutions. Then $\psi = \phi_1 - \phi_2$ obeys the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi,$$

and

$$\psi|_{\partial\Omega\times[0,\infty)} = \psi|_{\Omega\times\{0\}} = \left.\frac{\partial\psi}{\partial t}\right|_{\Omega\times\{0\}} = 0.$$

We let

$$E_\psi(t) = \frac{1}{2}\int_\Omega \left[\left(\frac{\partial\psi}{\partial t}\right)^2 + c^2\nabla\psi\cdot\nabla\psi\right]\,\mathrm{d}V.$$

Then since ψ obeys the wave equation with fixed boundary conditions, we know E_ψ is constant.

Initially, at $t = 0$, we know that $\psi = \frac{\partial\psi}{\partial t} = 0$. So $E_\psi(0) = 0$. At time t, we have

$$E_\psi = \frac{1}{2}\int_\Omega \left(\frac{\partial\psi}{\partial t}\right)^2 + c^2(\nabla\psi)\cdot(\nabla\psi)\,\mathrm{d}V = 0.$$

Hence we must have $\frac{\partial\psi}{\partial t} = 0$. So ψ is constant. Since it is 0 at the beginning, it is always 0. □

5 Distributions

5.1 Distributions

5.2 Green's functions

5.3 Green's functions for initial value problems

6 Fourier transforms

6.1 The Fourier transform

6.2 The Fourier inversion theorem

6.3 Parseval's theorem for Fourier transforms

Theorem (Parseval's theorem (again))**.** Suppose $f, g : \mathbb{R} \to \mathbb{C}$ are sufficiently well-behaved that $\tilde{f}$ and $\tilde{g}$ exist and we indeed have $\mathcal{F}^{-1}[\tilde{f}] = f, \mathcal{F}^{-1}[\tilde{g}] = g$. Then

$$(f, g) = \int_{\mathbb{R}} f^*(x) g(x)\;\mathrm{d}x = \frac{1}{2\pi}(\tilde{f}, \tilde{g}).$$

In particular, if $f = g$, then

$$\|f\|^2 = \frac{1}{2\pi}\|\tilde{f}\|^2.$$

Proof.

$$\begin{aligned}
(f, g) &= \int_{\mathbb{R}} f^*(x) g(x)\;\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f^*(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{g}(x)\;\mathrm{d}k\right]\;\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f^*(x) e^{ikx}\;\mathrm{d}x\right]\tilde{g}(k)\;\mathrm{d}k \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x) e^{-ikx}\;\mathrm{d}x\right]^*\tilde{g}(k)\;\mathrm{d}k \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}^*(k)\tilde{g}(k)\;\mathrm{d}k \\
&= \frac{1}{2\pi}(\tilde{f}, \tilde{g}).
\end{aligned}$$

□

6.4 A word of warning

6.5 Fourier transformation of distributions

6.6 Linear systems and response functions

6.7 General form of transfer function

6.8 The discrete Fourier transform

6.9 The fast Fourier transform*

7 More partial differential equations

7.1 Well-posedness

7.2 Method of characteristics

7.3 Characteristics for 2nd order partial differential equations

7.4 Green's functions for PDEs on $\mathbb{R}^n$

7.5 Poisson's equation

Proposition (Green's first identity)**.**

$$\int_{\partial\Omega} \phi \mathbf{n} \cdot \nabla\psi \;\mathrm{d}S = \int_{\Omega} \phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)\;\mathrm{d}V.$$

Proposition (Green's second identity)**.**

$$\int_{\Omega} \phi\nabla^2\psi - \psi\nabla^2\phi\;\mathrm{d}V = \int_{\partial\Omega} \phi\mathbf{n}\cdot\nabla\psi - \psi\mathbf{n}\cdot\nabla\phi\;\mathrm{d}S.$$

Proposition (Green's third identity)**.**

$$\phi(\mathbf{y}) = \int_{\partial\Omega} \phi(\mathbf{n}\cdot\nabla G_3) - G_3(\mathbf{n}\cdot\nabla\phi)\;\mathrm{d}S - \int_{\Omega} G_3(\mathbf{x},\mathbf{y})F(\mathbf{x})\;\mathrm{d}^3 x.$$