*Course C10—No. B6*

# METHODS — EXAMPLES I

## Fourier series

**1.** *Fourier coefficients (full-range series).* For the periodic function $f(\theta) = (\theta^2 - \pi^2)^2$ on the interval $-\pi \le \theta < \pi$, show that it has the Fourier series

$$f(\theta) = \frac{8\pi^4}{15} + 24 \sum_{n\neq 0} \frac{(-1)^{n+1}}{n^4} \mathrm{e}^{in\theta}\,.$$

[Remark: if you're happy that you can do the integrals you might like to save time by using www.integrals.com to evaluate them.] Sketch the function $f(\theta)$ and comment on its differentiability and the order of the terms in its Fourier series as $n \to \infty$.

**2.** *Fourier coefficients (half-range series).* Suppose that $f(\theta) = \theta^2$ for $0 \le \theta \le \pi$. Express $f(\theta)$ as (a) a Fourier sine series, and (b) a cosine series, each having period $2\pi$. Sketch the functions represented by (a) and (b) in the range $-6\pi$ to $6\pi$. If the series (a) and (b) are differentiated term-by-term, how are the answers related (if at all) to the Fourier series for $g(\theta) = 2\theta$ and $h(\theta) = 2|\theta|$ each in the range $(-\pi, \pi)$?

**3.** *Series summation.* Find the (complex) Fourier series of $f(\theta) = e^\theta$ for $\theta \in [-\pi, \pi)$. Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}(\pi \coth \pi - 1) \quad .$$

**4.** *Parseval's identity and a low pass filter.* (i) Given that a function $f(t)$ defined over the interval $(-T, T)$ has the Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right)\right], \qquad \text{show directly that} \qquad \frac{1}{T}\int_{-T}^{T} [f(t)]^2 dt = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\,,$$

where you may assume $f(t)$ is such that this series is convergent.

(ii) A unit amplitude square wave of period $2T$ is given by $f(t) = 1$ for $0 < t < T$ and $f(t) = -1$, for $-T < t < 0$. Suppose this is the input for a system which permits angular frequencies less than $\frac{9}{2}\pi T^{-1}$ to be perfectly transmitted and frequencies greater than $\frac{9}{2}\pi T^{-1}$ to be perfectly absorbed. Calculate the form of the output. The power is proportional to the mean value of $f^2(t)$; what fraction of the incident power is transmitted?

**5.** *Discontinuities and the Wilbraham-Gibbs phenomenon*.* (i) Suppose that $f$ is a square wave given by

$$f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi\,. \end{cases} \qquad \text{Sketch } f \text{ and show that} \qquad f(\theta) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)\theta}{2k-1}\,.$$

(ii) Now define the partial sum of this series as

$$S_n(\theta) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{n} \frac{\sin(2k-1)\theta}{2k-1}\,,$$

and find the following expression

$$S_n(\theta) = \frac{1}{2} + \frac{1}{\pi}\int_0^\theta \frac{\sin 2nt}{\sin t}\mathrm{d}t\,.$$

[Hint: consider $S_n'(\theta)$ for the two forms.]

(iii) Deduce that $S_n(\theta)$ has extrema at $\theta = m\pi/2n$, $n = 1, 2, \ldots 2n-1, 2n+1, \ldots$, (*i.e.*, all integer $m$ except even multiples of $n$) and that the height of the first maximum for large $n$ is approximately

$$S_n\left(\frac{\pi}{2n}\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\pi \frac{\sin u}{u}\,\mathrm{d}u \ (\simeq\ 1{\cdot}089)\,.)$$

Comment on the accuracy of Fourier series at discontinuities. (This question takes you through some important steps which are used in the proof of Fourier's theorem – refer, for example, to chapter 14 of Jeffreys & Jeffreys.)

## Sturm-Liouville theory

**6.** Eigenfunctions and eigenvalues. In the boundary value problem

$$y'' + \lambda y = 0;\ y(0) = 0, \quad y(1) + y'(1) = 0,$$

show that the eigenvalue $\lambda$ can take infinitely many values $\lambda_1 < \lambda_2 < \lambda_3$... Indicate roughly the behaviour of $\lambda_n$ as $n \to \infty$.

**7.** Recasting in Sturm-Liouville form. Express the following equations in Sturm-Liouville form:

$$\text{(i)}\ \ (1-x^2)y'' - 2xy' + n(n+1)y = 0\,, \qquad \text{(ii)}\ \ xy'' + (b-x)y' - ay = 0\,,$$

where $n, a$, and $b$ are constants.

(iii) Find the eigenvalues and eigenfunctions for

$$y'' + 4y' + (4+\lambda)y = 0,\ y(0) = y(1) = 0.$$

What is the orthogonality relation for these eigenfunctions?

**8.** Bessel's equation. (i) Show that the eigenvalues of the Sturm-Liouville problem

$$\frac{d}{dx}(x\frac{du}{dx}) + \lambda x u = 0\,, \qquad 0 < x < 1\,,$$

with $u(x)$ bounded as $x \to 0$ and $u(1) = 0$, are $\lambda = j_n^2$ $(n = 1, 2, ...)$, where the $j_n$ are the zeros of the Bessel function $J_0(z)$, arranged in ascending order. [Recall: Bessel's equation of order zero is $\frac{d}{dz}(z\frac{dy}{dz}) + zy = 0\,, (z > 0)$, which you may assume has one solution $J_0(z)$ which is regular at $z = 0$ and a second linearly independent solution $Y_0(z)$ which is singular at $z = 0$.]

(ii) Using integration by parts on the differential equations for $J_0(\alpha x)$ and $J_0(\beta x)$, show that

$$\int_0^1 J_0(\alpha x)J_0(\beta x)x dx = \frac{\beta J_0(\alpha)J_0'(\beta) - \alpha J_0(\beta)J_0'(\alpha)}{\alpha^2 - \beta^2} \quad (\beta \neq \alpha)$$

$$\int_0^1 J_0(j_n x)J_0(j_m x)x dx = 0\,, \quad (n \neq m)\,, \qquad \int_0^1 [J_0(j_n x)]^2 x dx = \frac{1}{2}[J_0'(j_n)]^2. \quad [\textit{Hint:}\ \text{Consider}\ \beta = j_n + \epsilon\ \text{as}\ \epsilon \to 0.]$$

(iii) Assume that the inhomogeneous equation

$$\frac{d}{dx}(x\frac{du}{dx}) + \tilde{\lambda} x u = x f(x),$$

where $\tilde{\lambda}$ is not an eigenvalue, has a unique solution such that $u(x)$ is bounded as $x \to 0$ and $u(1) = 0$. Assuming also that $f(x)$ satisfies the same boundary conditions as $u$ and the completeness of the eigenfunctions $J_0(j_n x)$, obtain the eigenfunction expansion of $u$.

**9.** Higher order self-adjoint form*. Consider the fourth-order differential operator

$$\mathcal{L} = \sum_{r=0}^{4} p_r(x)\frac{d^r}{dx^r},$$

where the $p_r(x)$ are real functions, with BCs $y(0) = y(1) = y'(0) = y'(1) = 0$. Show that $\mathcal{L}$ is self-adjoint if and only if $p_3 = 2p_4',\ p_1 = p_2' - p'''_4$.

Considering a specific example, show that the boundary value problem

$$-y'''' + \lambda y = 0;\ y(0) = y(1) = y'(0) = y'(1) = 0$$

has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$... . Indicate roughly the behaviour of $\lambda_n$ as $n \to \infty$.

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# METHODS — EXAMPLES II

## The one-dimensional wave equation

**1.** *Modes on a string*. A uniform string of line density $\mu$ and tension $T$ undergoes small transverse vibrations of amplitude $y(x,t)$. The string is fixed at $x=0$ and $x=\ell$, and satisfies the initial conditions

$$y(x,0)=0\,, \qquad y_t(x,0)=\frac{4V}{\ell^2}x(\ell-x)\,, \quad \text{for} \quad 0<x<\ell\,,$$

where $y_t \equiv \partial y/\partial t$. Using the fact that $y(x,t)$ is a solution of the wave equation, find the amplitudes of the normal modes and deduce the kinetic and potential energies of the string at time $t$. By comparison with the initial energy of the string show that

$$\sum_{\text{n odd}} \frac{1}{n^6} = \frac{\pi^6}{960}\,.$$

**2.** *Damped string motion*. (i) A uniform stretched string of length $\ell$, density per unit length $\mu$ and tension $T=\mu c^2$ is fixed at both ends. The motion of the string is resisted by the surrounding medium, the resistive force on unit length being $-2k\mu y_t$, where $y$ is the transverse displacement and the constant $k=\pi c/\ell$. Show that the equation of motion of the string is

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} + 2k\frac{\partial y}{\partial t}\,,$$

and find $y(x,t)$ given that $y(x,0)=A\sin(\pi x/\ell)$ and $y_t(x,0)=0$.

(ii) If an extra transverse force $F_0\sin(\pi x/\ell)\cos(\pi ct/\ell)$ per unit length acts on the string, find the resulting forced oscillation. [*Hint:* You may find it useful to guess a particular solution to combine with the general homogeneous solution that you probably derived in (i).]

**3.** *Wave reflection and transmission*. A string of uniform density is stretched along the $x$-axis under tension $T$ and undergoes small transverse oscillations in the $(x,y)$ plane so that its displacement $y(x,t)$ satisfies

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}\,, \qquad (*)$$

where $c$ is a constant.

(i) Show that if a mass $M$ is fixed to the string at $x=0$ then its equation of motion can be written

$$M\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=0} = T\left[\frac{\partial y}{\partial x}\right]_{x=0_-}^{x=0_+}\,.$$

(ii) Suppose that a wave $\exp[i\omega(t-x/c)]$ is incident from $x=-\infty$. Obtain the amplitudes and phases of the reflected and transmitted waves, and comment on their values when $\lambda = M\omega c/T$ is large or small.

**4.** *Impulsive force on a string*. The displacement $y(x,t)$ of a uniform string stretched between the points $x=0,\ell$ satisfies the wave equation $(*)$ given above with the boundary conditions, $y(0,t)=y(\ell,t)=0$. For $t<0$ the string oscillates in its fundamental mode and $y(x,0)=0$. A musician strikes the midpoint of the string impulsively at time $t=0$ so that the change in $\partial y/\partial t$ at $t=0$ is $\lambda\delta(x-\frac{1}{2}\ell)$. Find $y(x,t)$ for $t>0$.

## Laplace's equation

**5.** *Cartesian coordinates*. Show that the solution of $\nabla^2\phi=0$ in the region $0<x<a, 0<y<b, 0<z<c$, with $\phi=1$ on the surface $z=0$ and $\phi=0$ on all the other surfaces is

$$\phi = \frac{16}{\pi^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\sinh[\ell(c-z)]\sin[(2p+1)\pi x/a]\sin[(2q+1)\pi y/b]}{(2p+1)(2q+1)\sinh c\ell}\,,$$

where $\ell^2=(2p+1)^2\pi^2/a^2+(2q+1)^2\pi^2/b^2$. [*Hint:* You may find it useful to use the above form of the $z$-dependent part of the solution from the outset.] Discuss the behaviour of the solutions as $c\to\infty$.

**6.** *Plane polar coordinates.* The potential $\phi$ satisfies Laplace's equation in the unit circle $r < 1$ with boundary condition

$$\phi(r=1,\theta) = \begin{cases} \pi/2\,, & 0 \leq \theta < \pi\,. \\ -\pi/2\,, & \pi \leq \theta < 2\pi\,. \end{cases}$$

Using the method of separation of variables show that

$$\phi(r,\theta) = 2 \sum_{n\ odd} \frac{r^n \sin n\theta}{n}\,.$$

Sum the series using the substitution $z = re^{i\theta}$. [Your solution can then be interpreted geometrically in terms of the angle between the lines to the two points on the boundary where the data jumps.]

## Legendre polynomials

**7.** *Eigenfunction derivatives.* If $y_m$ and $y_n$ are real eigenfunctions of the Sturm-Liouville equation

$$\frac{d}{dx}(p(x)\frac{dy}{dx}) + (\lambda - q(x))y = 0\,,\ (a < x < b)\,,\ \text{ satisfying the normalisation condition } \int_a^b y_m^2 dx = \int_a^b y_n^2 dx = 1\,,$$

show that (under suitable boundary conditions)

$$\int_a^b (py_m^{'}\, y_n^{'} + qy_m y_n)dx = \lambda_m \delta_{mn} \ \ \text{(no summation)}.$$

With $P_n$ a Legendre polynomial, use this result to evaluate $\displaystyle\int_{-1}^{1} (1-x^2)P_m^{'}(x)P_n^{'}(x)dx\,.$

**8.** *Legendre polynomials and multipole moments.* Show that $1/|\mathbf{r}-\mathbf{k}|$ obeys Laplace's equation in three dimensions whenever $\mathbf{r} \neq \mathbf{k}$. Taking $\mathbf{k}$ to be a unit vector in the $z$-direction, show that

$$P_\ell(x) = \frac{1}{\ell!}\frac{d^\ell}{dr^\ell}\frac{1}{\sqrt{1-2r\cos\theta + r^2}}\bigg|_{r=0}$$

by expanding this solution of Laplace's equation in the region $r < 1$. Use the integral

$$\int_{-1}^{1} \frac{1}{1-2rx+r^2}\,dx$$

to show that the Legendre polynomials obey the normalization condition $\int_{-1}^{1} P_\ell(x)^2 dx = 2/(2\ell+1)$. Show also that $P'_{\ell+1}(x) - P'_{\ell-1}(x) = (2\ell+1)P_\ell(x)$.

**9.** *Spherical polar coordinates.* You've just shown that the electrostatic potential in a charge-free region satisfies Laplace's equation. Find the potential inside a spherical region bounded by two (conducting) hemispheres upon which the potential takes the values $\pm V$ respectively. [*Hint:* Note that $\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$.]

## The heat equation

**10.** *Diffusion in a disc & Bessel functions.* Consider the unit disc, with initial temperature distribution $\psi_0(r,\theta)$. Require the boundary of the disc to be held at (wlog) zero temperature $\psi(1,\theta,t) = 0$ for all $t > 0$. By assuming that the temperature satisfies the diffusion equation in the disc (with unit diffusion coefficient) show that the solution is

$$\psi = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty} c_{nk} J_n(j_{nk}r)\exp[in\theta - j_{nk}^2 t],$$

where $j_{nk}$ is the $k^{\text{th}}$ smallest (positive) zero of the $n^{\text{th}}$ order Bessel function of the first kind, (i.e. $J_n(j_{nk}) = 0$) and present an appropriate expression for $c_{nk}$, showing explicitly that

$$\int_0^1 J_n(j_{nk}r)J_n(j_{nl}r)rdr = \frac{\delta_{kl}[J_n'(j_{nk})]^2}{2} = \frac{\delta_{kl}J_{n+1}^2(j_{nk})}{2}.$$

Suppose now that the initial temperature $\psi_0(r,\theta) = \Psi_0$ is constant. Show that the only non-zero coefficients have $n = 0$, and are equal to

$$c_{0k} = \frac{2\Psi_0}{j_{0k}J_1(j_{0k})}.$$

What is the spatial structure of the temperature distribution as $t \to \infty$?
[*The recursion relations* $[z^{-\nu}J_\nu(z)]' = -z^{-\nu}J_{\nu+1}(z)$ *and* $[z^{\nu+1}J_{\nu+1}(z)]' = z^{\nu+1}J_\nu(z)$ *may be useful, as is the fact that Q8 of the first example sheet can be generalized straightforwardly to* $J_n$ *for arbitrary positive integers* $n$.]

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# METHODS — EXAMPLES III

## Green's functions

**1.** *Initial value problem*. The reading $\theta(t)$ of an ammeter satisfies

$$\ddot{\theta} + 2p\dot{\theta} + (p^2 + q^2)\theta = f(t)\,,$$

where $p, q$ are constants with $p > 0$. The ammeter is set so that $\theta$ and $\dot{\theta}$ are zero when $t = 0$. Assuming $q \neq 0$, show by constructing the Green's function that

$$\theta(t) = \frac{1}{q}\int_0^t e^{-p(t-\tau)}\sin[q(t-\tau)]f(\tau)d\tau\,.$$

Derive the same result using Fourier transforms, showing that the transfer function for this system is

$$\tilde{R}(\omega) = \frac{1}{2qi}\left[\frac{1}{(i\omega + p - qi)} - \frac{1}{(i\omega + p + qi)}\right].$$

**2.** *Boundary value problem*. Obtain the Green's function $G(x, \xi)$ satisfying

$$\frac{d^2G}{dx^2} - \lambda^2 G = \delta(x - \xi)\,, \qquad 0 \le x \le 1\,, \quad 0 \le \xi \le 1\,.$$

where $\lambda$ is real, subject to the boundary conditions $G(0, \xi) = G(1, \xi) = 0$. Show that the solution to the equation

$$\frac{d^2y}{dx^2} - \lambda^2 y = f(x), \qquad \text{subject to the same boundary conditions is}$$

$$y = -\frac{1}{\lambda\sinh\lambda}\left\{\sinh\lambda x\int_x^1 f(\xi)\sinh\lambda(1-\xi)d\xi + \sinh\lambda(1-x)\int_0^x f(\xi)\sinh\lambda\xi d\xi\right\}.$$

**3.** *Finite asymptotics*. A linear differential operator is defined by

$$L_x y = -\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + y\,.$$

By writing $y = z/x$ or otherwise, find those solutions of $L_x y = 0$ which are either (a) bounded as $x \to 0$, or (b) bounded as $x \to \infty$. Find the Green's function $G(x, a)$ satisfying

$$L_x G(x, a) = \delta(x - a)\,,$$

and both conditions (a) and (b). Use $G(x, a)$ to solve (subject to conditions (a) and (b))

$$L_x y(x) = \begin{cases} 1, & \text{for } 0 \le x \le R\,, \\ 0, & \text{for } x > R\,. \end{cases}$$

Show that the solution has the form, for suitable constants $A, B$

$$y(x) = \begin{cases} 1 + Ax^{-1}\sinh x\,, & \text{for } 0 \le x \le R\,, \\ Bx^{-1}e^{-x}\,, & \text{for } x > R\,. \end{cases}$$

**4.** *Higher order initial value problem**. Show that the Green's function for the initial value problem ($' \equiv \frac{d}{dt}$)

$$y'''' + k^2 y'' = f(t)\,, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0\,,$$

$$\text{is given by} \qquad G(t, \tau) = \begin{cases} 0\,, & 0 < t < \tau\,, \\ k^{-2}(t-\tau) - k^{-3}\sin k(t-\tau)\,, & t > \tau\,. \end{cases}$$

Hence find the solution for $f(t) = e^{-t}$.
[*Hint:* To make the calculations easier, for $t > \tau$ write the general homogeneous solution as a function of $t - \tau$.]

## The Dirac delta function

**5.** *Delta function properties*. The function $\phi(x)$ is monotone increasing in $[a,b]$ and has a (simple) zero at $x=c$ (i.e. $\phi'(c)\neq 0$) where $a<c<b$. Show that

$$\int_a^b f(x)\delta[\phi(x)]dx = \frac{f(c)}{|\phi'(c)|}\ .$$

Show that the same formula applies if $\phi(x)$ is monotone decreasing and hence derive a formula for general $\phi(x)$ provided the zeros are simple. Deduce that $\delta(at)=\delta(t)/|a|$ for $a\neq 0$. Also establish that

$$\int_{-\infty}^{+\infty} |x|\delta(x^2-a^2)dx = 1 \quad .$$

**6.** *Delta function derivative**. Show using polar coordinates that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x^2+y^2)\delta'(x^2+y^2-1)\delta(x^2-y^2)dx\,dy = f(1)-f'(1)$$

(where you may assume that $f(r^2)/r$ has a finite limit at $r=0$).

## Fourier transforms

**7.** *Duality property of FTs*. (a) Use the duality property of $FT$s (i.e. that $FT(FT(f))(x)=2\pi f(-x)$) to derive the frequency shift property from the translation property. (b) Let $h(x)=f(x)g(x)$. Starting with the convolution theorem (that $FT$ of a convolution is the product of individual $FT$s) show that $\tilde{h}(k)=\tilde{f}*\tilde{g}(k)/2\pi$.

**8.** *Functions with discontinuities*. Let $f(x)=e^{-x}$ for $0<x<\infty$, and $f(x)=0$ for $x<0$. Show that $\tilde{f}(k)=\frac{1-ik}{1+k^2}$. Show that the inverse Fourier transform of this Fourier transform $\tilde{f}(k)$ takes the value of $1/2$ at $x=0$. (This is a general property of Fourier transforms, analogously to Fourier series. Inversion for general $x$ is really straightforward with Complex Methods.)

**9.** *Fourier transform of Gaussians*. By using differentiation and the shift property, calculate the Fourier transform of a Gaussian distribution with a peak at $\mu\neq 0$, i.e. $f(x)=\exp[-n^2(x-\mu)^2]$.
Now let $\mu=0$, and consider $\delta_n(x)=(n/\sqrt{\pi})f(x)$. Sketch $\delta_n(x)$ and $\tilde{\delta}_n(k)$ for small and large $n$. What is $\int_{-\infty}^{\infty}\delta_n(x)dx$? What is happening as $n\to\infty$?

**10.** *Fast Fourier transform for DFT*. Consider $DFT_N$ the discrete Fourier transform mod $N$ with $N=2^m$ being a power of 2.
(a) For $\underline{a}=(a_0,\ldots,a_{N-1})$ show that direct computation of $DFT_N(\underline{a})$ by matrix multiplication requires $2N^2-N$ basic multiplication and addition operations between the matrix elements of $DFT_N$ and the components of $\underline{a}$.
(b) Show that $DFT_N(\underline{a})$ can be expressed in terms of two applications of $DFT_{N/2}$. (Hint: consider separately the even and odd numbered components of $\underline{a}$). Using this decomposition show that $DFT_N$ may be computed with $T(N)$ basic additions and multiplications where $T(N)$ has leading term $N\log_2 N$ i.e. exponentially faster as a function of $m$ than the direct method of (a). Find the exact formula for $T(N)$.

**11.** *Parseval's relation*. By considering the the Fourier transform of the function $f(x)=\cos(x)$ for $|x|<\pi/2$ and $f(x)=0$ for $|x|\geq\pi/2$, and the Fourier transform of its derivative, show that

$$\int_0^\infty \frac{\frac{\pi^2}{4}\cos^2 t}{\left(\frac{\pi^2}{4}-t^2\right)^2}dt = \int_0^\infty \frac{t^2\cos^2 t}{\left(\frac{\pi^2}{4}-t^2\right)^2}dt = \frac{\pi}{4}.$$

**12.** *Laplace's equation*. Show that the inverse Fourier transform of the function (for any real $\alpha$)

$$\tilde{f}_\alpha(k) = \begin{cases} e^{k\alpha}-e^{-k\alpha}, & |k|\leq 1,\\ 0 & |k|>1, \end{cases}$$

is

$$f_\alpha(x) = \frac{2i}{\pi(\alpha^2+x^2)}(\alpha\cosh\alpha\sin x - x\cos x\sinh\alpha).$$

Determine, by using Fourier transforms, the solution of Laplace's equation in the infinite strip $0\leq y\leq 1$, i.e.

$$\nabla^2\psi = 0;\quad -\infty<x<\infty,\ 0<y<1,$$

where $\psi(x,0)=f_1(x)$ the function given above with $\alpha=1$, and $\psi(x,1)=0$ for $-\infty<x<\infty$.

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# METHODS — EXAMPLES IV



## General properties of PDEs

**1.** *Characteristics*.
i) Find the characteristic curves of $u_x + yu_y = 0$. Hence find the solution of the problem with the boundary data $u(0, y) = y^3$.
ii) Solve for $u$ which satisfies $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$. In which region of the plane is the solution uniquely determined?
iii) Find $u$ such that $u_x + u_y + u = e^{x+2y}$, and $u(x, 0) = 0$.

**2.** *Well-posedness*.
The **backward** diffusion equation may be defined as

$$u_{xx} + u_t = 0.$$

Consider a domain $0 < x < \pi$, with $u(0, t) = 0 = u(\pi, t)$, and $u(x, 0) = U(x)$. By using the method of separation of variables, show that the problem is not well-posed. [*It may be helpful to scale the eigenfunctions you calculate similarly to the example in the lectures.*]

**3.** *Classification*.
i) Determine the regions where Tricomi's equation

$$u_{xx} + xu_{yy} = 0,$$

is of elliptic, parabolic and hyperbolic types. Derive its characteristics and canonical form in the hyperbolic region.
ii) Reduce the equation

$$u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0,$$

to the simple canonical form $u_{\xi\eta} = 0$ in its hyperbolic region, and hence show that

$$u = f(x + 2[-y]^{1/2}) + g(x - 2[-y]^{1/2}),$$

where $f$ and $g$ are arbitrary functions.

## Properties of Green's functions

**4.** *Symmetry*.
Consider a Dirichlet Green's function $G(\mathbf{r}; \mathbf{r}_0)$ for the Laplacian defined in an arbitrary three-dimensional domain $\mathcal{D}$. By using Green's second identity, show that $G(\mathbf{r}; \mathbf{r}_0) = G(\mathbf{r}_0; \mathbf{r})$ for all $\mathbf{r} \neq \mathbf{r}_0$ in the domain $\mathcal{D}$.

**5.** *Representation formula in 2D*.
If $u$ is a harmonic function in a 2D domain $\mathcal{D}$, with boundary $\delta\mathcal{D}$, show that

$$u(\mathbf{x}_0) = \frac{1}{2\pi} \oint_{\delta\mathcal{D}} \left[ u(\mathbf{x}) \frac{\partial}{\partial n}(\log|\mathbf{x} - \mathbf{x}_0|) - \log|\mathbf{x} - \mathbf{x}_0| \frac{\partial u}{\partial n} \right] dl,$$

where $dl$ is an arc element of $\delta\mathcal{D}$, $\mathbf{x} \in \delta\mathcal{D}$, $\mathbf{x}_0 \in \mathcal{D}$.

## Applications of Green's functions

**6.** *Cauchy problem in the half-plane for the Laplacian*.
Consider Laplace's equation in the half-plane with prescribed boundary conditions at $y = 0$, i.e.

$$\nabla^2\psi = 0;\ -\infty < x < \infty,\ y \geq 0,$$

where $\psi(x, 0) = f(x)$ a known function, such that $\psi$ tends to zero as $y \to \infty$.
i) Find the Green's function for this problem.
ii) Hence show that the solution is given by Poisson's integral formula:

$$\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x - \xi)^2 + y^2} d\xi.$$

iii) Derive the same result by taking Fourier transforms with respect to $x$ (assuming all transforms exist).
iv) Find (in closed form) and sketch the solution for various $y > 0$ when $f(x) = \psi_0$, $|x| < a$, and $f(x) = 0$ otherwise. Sketch the solution along $x = \pm a$.

**7.** _Wave equation._
An infinite string, at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\delta(x - x_0)$, where $V$ is a constant. Derive the position of the string for $t > 0$.

**8.** _Wave equation: Method of images._
A semi-infinite string, fixed for all time at zero at $x = 0$ and at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\delta(x - x_0)$, where $V$ is a constant. Derive the position of the string for $t > 0$, and compare the solution to the infinite case in the previous question.

**9.** _Diffusion equation with a boundary source._
Consider the problem on the half-line:

$$\theta_t - D\theta_{xx} = f(x,t), \qquad 0 < x < \infty, \qquad 0 < t < \infty,$$

with boundary and initial data $\theta(0,t) = h(t)$, $\theta(x,0) = \Theta(x)$. By considering the variable $V(x,t) = \theta(x,t) - h(t)$, and using the method of images, derive the general solution.

**10.** _Dirichlet Green's function for the sphere*._
i) Show that the Dirichlet Green's function for the Laplacian for the **interior** of a spherical domain of radius $a$ is

$$G(\mathbf{x};\mathbf{x}_0) = \frac{-1}{4\pi|\mathbf{x}-\mathbf{x}_0|} + \frac{a}{|\mathbf{x}_0|}\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0^*|}, \qquad \text{where} \qquad \mathbf{x}_0^* = \frac{a^2\mathbf{x}_0}{|\mathbf{x}_0|^2}.$$

ii) Derive the Dirichlet Green's function for the Laplacian for the **exterior** of a spherical domain of radius $a$.

**11.** _Forced wave equation._
Consider the forced wave equation with zero initial conditions

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Verify directly that

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds\,,$$

and hence determine the appropriate Green's function for the wave equation satisfying

$$\frac{\partial^2}{\partial t^2}G(x,t;\xi,\tau) - c^2\frac{\partial^2}{\partial x^2}G(x,t;\xi,\tau) = \delta(x-\xi)\,\delta(t-\tau),$$

$$G(x,0;\xi,\tau) = 0, \qquad \frac{\partial}{\partial t}G(x,0;\xi,\tau) = 0.$$

Calculate $u(x,t)$ explicitly in the case where $f(x,t) = \cos x$ and hence determine the times when $u = 0$ for all values of $x$.