Part IB — Methods
Definitions

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Self-adjoint ODEs
Periodic functions. Fourier series: definition and simple properties; Parseval's theorem. Equations of second order. Self-adjoint differential operators. The Sturm-Liouville equation; eigenfunctions and eigenvalues; reality of eigenvalues and orthogonality of eigenfunctions; eigenfunction expansions (Fourier series as prototype), approximation in mean square, statement of completeness. [5]

PDEs on bounded domains: separation of variables
Physical basis of Laplace's equation, the wave equation and the diffusion equation. General method of separation of variables in Cartesian, cylindrical and spherical coordinates. Legendre's equation: derivation, solutions including explicit forms of P_0, P_1 and P_2, orthogonality. Bessel's equation of integer order as an example of a self-adjoint eigenvalue problem with non-trivial weight.

Examples including potentials on rectangular and circular domains and on a spherical domain (axisymmetric case only), waves on a finite string and heat flow down a semi-infinite rod. [5]

Inhomogeneous ODEs: Green's functions
Properties of the Dirac delta function. Initial value problems and forced problems with two fixed end points; solution using Green's functions. Eigenfunction expansions of the delta function and Green's functions. [4]

Fourier transforms
Fourier transforms: definition and simple properties; inversion and convolution theorems. The discrete Fourier transform. Examples of application to linear systems. Relationship of transfer function to Green's function for initial value problems. [4]

PDEs on unbounded domains
Classification of PDEs in two independent variables. Well posedness. Solution by the method of characteristics. Green's functions for PDEs in 1, 2 and 3 independent variables; fundamental solutions of the wave equation, Laplace's equation and the diffusion equation. The method of images. Application to the forced wave equation, Poisson's equation and forced diffusion equation. Transient solutions of diffusion problems: the error function. [6]

Contents

0 Introduction

1 Vector spaces

Definition (Vector space)**.** A *vector space* over $\mathbb{C}$ (or $\mathbb{R}$) is a set V with an operation $+$ which obeys

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity)

(iii) There is some $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ (identity)

We can also multiply vectors by a scalars $\lambda \in \mathbb{C}$, which satisfies

(i) $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$ (associativity)

(ii) $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$ (distributivity in V)

(iii) $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \lambda\mathbf{v}$ (distributivity in $\mathbb{C}$)

(iv) $1\mathbf{v} = \mathbf{v}$ (identity)

Definition (Inner product)**.** An *inner product* on V is a map $(\,\cdot\,,\,\cdot\,) : V \times V \to \mathbb{C}$ that satisfies

(i) $(\mathbf{u}, \lambda\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v})$ (linearity in second argument)

(ii) $(\mathbf{u}, \mathbf{v} + \mathbf{w}) = (\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{w})$ (additivity)

(iii) $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})^*$ (conjugate symmetry)

(iv) $(\mathbf{u}, \mathbf{u}) \geq 0$, with equality iff $\mathbf{u} = \mathbf{0}$ (positivity)

Note that the positivity condition makes sense since conjugate symmetry entails that $(\mathbf{u}, \mathbf{u}) \in \mathbb{R}$.

The inner product in turn defines a norm $\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}$ that provides the notion of length and distance.

Definition (Basis)**.** A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ form a *basis* of V iff any $\mathbf{u} \in V$ can be uniquely written as a linear combination

$$\mathbf{u} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i$$

for some scalars λ_i. The *dimension* of a vector space is the number of basis vectors in its basis.

A basis is *orthogonal* (with respect to the inner product) if $(\mathbf{v}_i, \mathbf{v}_j) = 0$ whenever $i \neq j$.

A basis is *orthonormal* (with respect to the inner product) if it is orthogonal and $(\mathbf{v}_i, \mathbf{v}_i) = 1$ for all i.

Definition (Homogeneous boundary conditions)**.** A boundary condition is *homogeneous* if whenever f and g satisfy the boundary conditions, then so does $\lambda f + \mu g$ for any $\lambda, \mu \in \mathbb{C}$ (or $\mathbb{R}$).

2 Fourier series

Definition (Periodic function)**.** A function f is *periodic* if there is some fixed R such that $f(x+R) = f(x)$ for all x.

However, it is often much more convenient to think of this as a function $f : S^1 \to \mathbb{C}$ from unit circle to $\mathbb{C}$, and parametrize our function by an angle θ ranging from 0 to 2π.

2.1 Fourier series

2.2 Convergence of Fourier series

2.3 Differentiability and Fourier series

3 Sturm-Liouville Theory

3.1 Sturm-Liouville operators

Definition (Adjoint and self-adjoint)**.** The *adjoint* B of a map $A : V \to V$ is a map such that

$$(B\mathbf{u}, \mathbf{v}) = (\mathbf{u}, A\mathbf{v})$$

for all vectors $\mathbf{u}, \mathbf{v} \in V$. A map is then *self-adjoint* if

$$(M\mathbf{u}, \mathbf{v}) = (\mathbf{u}, M\mathbf{v}).$$

Definition (Inner product with weight)**.** An *inner product with weight* w, written $(\,\cdot\,,\,\cdot\,)_w$, is defined by

$$(f, g)_w = \int_a^b f^*(x)g(x)w(x)\;\mathrm{d}x,$$

where w is real, non-negative, and has only finitely many zeroes.

Definition (Eigenfunction with weight)**.** An *eigenfunction with weight* w of $\mathcal{L}$ is a function $y : [a, b] \to \mathbb{C}$ obeying the differential equation

$$\mathcal{L}y = \lambda w y,$$

where $\lambda \in \mathbb{C}$ is the eigenvalue.

3.2 Least squares approximation

4 Partial differential equations

4.1 Laplace's equation

Definition (Laplace's equation)**.** *Laplace's equation* on $\mathbb{R}^n$ says that a (twice-differentiable) equation $\phi : \mathbb{R}^n \to \mathbb{C}$ obeys

$$\nabla^2 \phi = 0,$$

where

$$\nabla^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Definition (Harmonic functions)**.** Functions that obey Laplace's equation are often called *harmonic functions*.

4.2 Laplace's equation in the unit disk in $\mathbb{R}^2$

4.3 Separation of variables

4.4 Laplace's equation in spherical polar coordinates

4.4.1 Laplace's equation in spherical polar coordinates

4.4.2 Legendre Polynomials

4.4.3 Solution to radial part

4.5 Multipole expansions for Laplace's equation

4.6 Laplace's equation in cylindrical coordinates

4.7 The heat equation

Definition (Heat equation)**.** The *heat equation* for a function $\phi : \Omega \to \mathbb{C}$ is

$$\frac{\partial \phi}{\partial t} = \kappa \nabla^2 \phi,$$

where $\kappa > 0$ is the *diffusion constant.*

4.8 The wave equation

5 Distributions

5.1 Distributions

Definition (Dirac-delta)**.** The *Dirac-delta* is a distribution defined by

$$\delta[\phi] = \phi(0).$$

5.2 Green's functions

5.3 Green's functions for initial value problems

6 Fourier transforms

6.1 The Fourier transform

Definition (Fourier transform)**.** The *Fourier transform* of an (absolutely integrable) function $f: \mathbb{R} \to \mathbb{C}$ is defined as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\;\mathrm{d}x$$

for *all* $k \in \mathbb{R}$. We will also write $\tilde{f}(k) = \mathcal{F}[f(x)]$.

6.2 The Fourier inversion theorem

6.3 Parseval's theorem for Fourier transforms

6.4 A word of warning

6.5 Fourier transformation of distributions

6.6 Linear systems and response functions

6.7 General form of transfer function

6.8 The discrete Fourier transform

6.9 The fast Fourier transform*

7 More partial differential equations

7.1 Well-posedness

Definition (Well-posed problem)**.** A partial differential equation problem is said to be well-posed if its Cauchy data means

(i) A solution exists;

(ii) The solution is unique;

(iii) A "small change" in the Cauchy data leads to a "small change" in the solution.

7.2 Method of characteristics

Definition (Tangent vector)**.** The tangent vector to a smooth curve C given by $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ with $\mathbf{x}(s) = (x(s), y(s))$ is

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right).$$

Definition (Integral curve)**.** Let $\mathbf{V}(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field. The *integral curves* associated to $\mathbf{V}$ are curves whose tangent $\left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right)$ is just $\mathbf{V}(x, y)$.

7.3 Characteristics for 2nd order partial differential equations

Definition (Symbol and principal part)**.** Let $\mathcal{L}$ be the general 2nd order differential operator on $\mathbb{R}^n$. We can write it as

$$\mathcal{L} = \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^n b^i(x) \frac{\partial}{\partial x^i} + c(X),$$

where $a^{ij}(x), b^i(x), c(x) \in \mathbb{R}$ and $a^{ij} = a^{ji}$ (wlog).

We define the *symbol* $\sigma(\mathbf{k}, x)$ of $\mathcal{L}$ to be

$$\sigma(\mathbf{k}, x) = \sum_{i,j=1}^n a^{ij}(x) k_i k_j + \sum_{i=1}^n b^i(x) k_i + c(x).$$

So we just replace the derivatives by the variable k.

The *principal part* of the symbol is the leading term

$$\sigma^p(\mathbf{k}, x) = \sum_{i,j=1}^n a^{ij}(x) k_i k_j.$$

Definition (Elliptic, hyperbolic, ultra-hyperbolic and parabolic differential operators)**.** Let $\mathcal{L}$ be a differential operator. We say $\mathcal{L}$ is

– *elliptic at x* if all eigenvalues of $A(x)$ have the same sign. Equivalently, if $\sigma^p(\,\cdot\,, x)$ is a definite quadratic form;

- *hyperbolic at x* if all but one eigenvalues of $A(x)$ have the same sign;
- *ultra-hyperbolic at x* if $A(x)$ has more than one eigenvalues of each sign;
- *parabolic at x* if $A(x)$ has a zero eigenvalue, i.e. $\sigma^p(\,\cdot\,, x)$ is degenerate.

We say $\mathcal{L}$ is elliptic if $\mathcal{L}$ is elliptic at all x, and similarly for the other terms.

Definition (Characteristic surface)**.** Given a differential operator $\mathcal{L}$, let

$$f(x^1, x^2, \cdots, x^n) = 0$$

define a surface $C \subseteq \mathbb{R}^n$. We say C is *characteristic* if

$$\sum_{i,j=1}^n a^{ij}(x) \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} = (\nabla f)^T A(\nabla f) = \sigma^p(\nabla f, x) = 0.$$

In the case where we only have two dimensions, a characteristic surface is just a curve.

7.4 Green's functions for PDEs on $\mathbb{R}^n$

7.5 Poisson's equation