Part IB — Markov Chains

Theorems

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Discrete-time chains

Definition and basic properties, the transition matrix. Calculation of n-step transition probabilities. Communicating classes, closed classes, absorption, irreducibility. Calculation of hitting probabilities and mean hitting times; survival probability for birth and death chains. Stopping times and statement of the strong Markov property. [5]

Recurrence and transience; equivalence of transience and summability of n-step transition probabilities; equivalence of recurrence and certainty of return. Recurrence as a class property, relation with closed classes. Simple random walks in dimensions one, two and three. [3]

Invariant distributions, statement of existence and uniqueness up to constant multiples. Mean return time, positive recurrence; equivalence of positive recurrence and the existence of an invariant distribution. Convergence to equilibrium for irreducible, positive recurrent, aperiodic chains *and proof by coupling*. Long-run proportion of time spent in a given state. [3]

Time reversal, detailed balance, reversibility, random walk on a graph. [1]

Contents

0 Introduction

1 Markov chains

1.1 The Markov property

Proposition.

(i) λ is a *distribution*, i.e. $\lambda_i \geq 0, \sum_i \lambda_i = 1$.

(ii) P is a *stochastic matrix*, i.e. $p_{i,j} \geq 0$ and $\sum_j p_{i,j} = 1$ for all i.

Theorem. Let λ be a distribution (on S) and P a stochastic matrix. The sequence $X = (X_0, X_1, \cdots)$ is a Markov chain with initial distribution λ and transition matrix P iff

$$\mathbb{P}(X_0 = i, X_1 = i_1, \cdots, X_n = i_n) = \lambda_{i_0} p_{i_0,i_1} p_{i_1,i_2} \cdots p_{i_{n-1},i_n} \qquad (*)$$

for all $n, i_0, \cdots, i_n$

Theorem (Extended Markov property)**.** Let X be a Markov chain. For $n \geq 0$, any H given in terms of the past $\{X_i : i < n\}$, and any F given in terms of the future $\{X_i : i > n\}$, we have

$$\mathbb{P}(F \mid X_n = i, H) = \mathbb{P}(F \mid X_n = i).$$

1.2 Transition probability

Theorem (Chapman-Kolmogorov equation)**.**

$$p_{i,j}(m + n) = \sum_{k \in S} p_{i,k}(m) p_{k,j}(n).$$

2 Classification of chains and states

2.1 Communicating classes

Proposition. $\leftrightarrow$ is an equivalence relation.

Proposition. A subset C is closed iff "$i \in C, i \to j$ implies $j \in C$".

2.2 Recurrence or transience

Theorem. i is recurrent iff $\sum_n p_{i,i}(n) = \infty$.

Theorem.

$$P_{i,j}(s) = \delta_{i,j} + F_{i,j}(s)P_{j,j}(s),$$

for $-1 < s \leq 1$.

Lemma (Abel's lemma)**.** Let $u_1, u_2, \cdots$ be real numbers such that $U(s) = \sum_n u_n s^n$ converges for $0 < s < 1$. Then

$$\lim_{s \to 1^-} U(s) = \sum_n u_n.$$

Theorem. i is recurrent iff $\sum_n p_{ii}(n) = \infty$.

Theorem. Let C be a communicating class. Then

(i) Either every state in C is recurrent, or every state is transient.

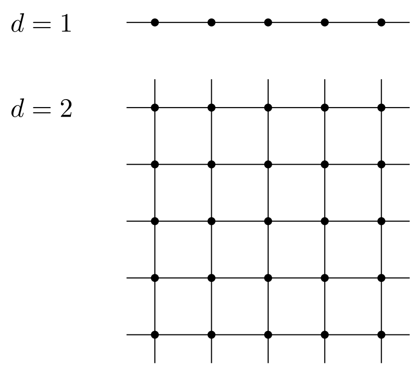
(ii) If C contains a recurrent state, then C is closed.

Theorem. In a finite state space,

(i) There exists at least one recurrent state.

(ii) If the chain is irreducible, every state is recurrent.

Theorem (Pólya's theorem)**.** Consider $\mathbb{Z}^d = \{(x_1, x_2, \cdots, x_d) : x_i \in \mathbb{Z}\}$. This generates a graph with x adjacent to y if $|x - y| = 1$, where $| \cdot |$ is the Euclidean norm.



Consider a random walk in $\mathbb{Z}^d$. At each step, it moves to a neighbour, each chosen with equal probability, i.e.

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} \frac{1}{2d} & |j - i| = 1 \\ 0 & \text{otherwise} \end{cases}$$

This is an irreducible chain, since it is possible to get from one point to any other point. So the points are either all recurrent or all transient.

The theorem says this is recurrent iff $d = 1$ or 2.

2.3 Hitting probabilities

Theorem. The vector $(h_i^A : i \in S)$ satisfies

$$h_i^A = \begin{cases} 1 & i \in A \\ \sum_{j \in S} p_{i,j} h_j^A & i \not\in A \end{cases},$$

and is *minimal* in that for any non-negative solution $(x_i : i \in S)$ to these equations, we have $h_i^A \leq x_i$ for all i.

Theorem. $(k_i^A : i \in S)$ is the minimal non-negative solution to

$$k_i^A = \begin{cases} 0 & i \in A \\ 1 + \sum_j p_{i,k} k_j^A & i \not\in A. \end{cases}$$

2.4 The strong Markov property and applications

Theorem (Strong Markov property)**.** Let X be a Markov chain with transition matrix P, and let T be a stopping time for X. Given $T < \infty$ and $X_T = i$, the chain $(X_{T+k} : k \geq 0)$ is a Markov chain with transition matrix P with initial distribution $X_{T+0} = i$, and this Markov chain is independent of $X_0, \cdots, X_T$.

2.5 Further classification of states

Theorem. Suppose $X_0 = i$. Let $V_i = |\{n \geq 1 : X_n = i\}|$. Let $f_{i,i} = \mathbb{P}_i(T_i < \infty)$. Then

$$\mathbb{P}_i(V_i = r) = f_{i,i}^r (1 - f_{i,i}),$$

since we have to return r times, each with probability $f_{i,i}$, and then never return.

Hence, if $f_{i,i} = 1$, then $\mathbb{P}_i(V_i = r) = 0$ for all r. So $\mathbb{P}_i(V_i = \infty) = 1$. Otherwise, $\mathbb{P}_i(V_i = r)$ is a genuine geometric distribution, and we get $\mathbb{P}_i(V_i < \infty) = 1$.

Theorem. If $i \leftrightarrow j$ are communicating, then

(i) $d_i = d_j$.

(ii) i is recurrent iff j is recurrent.

(iii) i is positive recurrent iff j is positive recurrent.

(iv) i is ergodic iff j is ergodic.

Proposition. If the chain is irreducible and $j \in S$ is recurrent, then

$$\mathbb{P}(X_n = j \text{ for some } n \geq 1) = 1,$$

regardless of the distribution of X_0.

3 Long-run behaviour

3.1 Invariant distributions

Theorem. Consider an irreducible Markov chain. Then

(i) There exists an invariant distribution if some state is positive recurrent.

(ii) If there is an invariant distribution π, then every state is positive recurrent, and
$$\pi_i = \frac{1}{\mu_i}$$
for $i \in S$, where μ_i is the mean recurrence time of i. In particular, π is unique.

Proposition. For an irreducible recurrent chain and $k \in S$, $\rho = (\rho_i : i \in S)$ defined as above by

$$\rho_i = \mathbb{E}_k(W_i), \quad W_i = \sum_{m=1}^{\infty} 1(X_m = i, T_k \geq m),$$

we have

(i) $\rho_k = 1$

(ii) $\sum_i \rho_i = \mu_k$

(iii) $\rho = \rho P$

(iv) $0 < \rho_i < \infty$ for all $i \in S$.

Theorem. Consider an irreducible Markov chain. Then

(i) There exists an invariant distribution if and only if some state is positive recurrent.

(ii) If there is an invariant distribution π, then every state is positive recurrent, and
$$\pi_i = \frac{1}{\mu_i}$$
for $i \in S$, where μ_i is the mean recurrence time of i. In particular, π is unique.

3.2 Convergence to equilibrium

Theorem. Consider a Markov chain that is irreducible, positive recurrent and aperiodic. Then
$$p_{i,k}(n) \to \pi_k$$
as $n \to \infty$, where π is the unique invariant distribution.

4 Time reversal

Theorem. Let X be positive recurrent, irreducible with invariant distribution π. Suppose that X_0 has distribution π. Then Y defined by

$$Y_k = X_{N-k}$$

is a Markov chain with transition matrix $\hat{P} = (\hat{p}_{i,j} : i, j \in S)$, where

$$\hat{p}_{i,j} = \left(\frac{\pi_j}{\pi_i}\right) p_{j,i}.$$

Also π is invariant for $\hat{P}$.

Proposition. Let P be the transition matrix of an irreducible Markov chain X. Suppose (P, λ) is in detailed balance. Then λ is the *unique* invariant distribution and the chain is reversible (when X_0 has distribution λ).