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MARKOV CHAINS

Example Sheet 1

1.1 Let $X = (X_n)$ be a Markov chain. Suppose that we are given that $X_m = i$. Show that $Z_k = X_{m+k}$, $k \geq 0$, is a Markov chain with starting state i.

1.2 Show that any sequence of independent random variables taking values in the countable set S is a Markov chain. Under what condition is this chain homogeneous?

1.3 Let X_n be the maximum reading obtained in the first n throws of a fair die. Show that X is a Markov chain, and find the transition probabilities $p_{ij}(n)$.

1.4 Let $\{S_n : n \geq 0\}$ be a simple random walk with $S_0 = 0$, and show that $X_n = |S_n|$ defines a Markov chain; find the transition probabilities of this chain. Let $M_n = \max\{S_k : 0 \leq k \leq n\}$, and show that $Y_n = M_n - S_n$ defines a Markov chain.

1.5 Let X be a Markov chain and let $\{n_r : r \geq 0\}$ be an unbounded increasing sequence of positive integers. Show that $Y_r = X_{n_r}$ constitutes a (possibly inhomogeneous) Markov chain. Find the transition matrix of Y when $n_r = 2r$ and X is a simple random walk.

1.6 Let X and Y be Markov chains on the set $\mathbb{Z}$ of integers. Is the sequence $Z_n = X_n + Y_n$ necessarily a Markov chain? Explain.

1.7 A flea hops about at random on the vertices of a triangle (i.e., each hop is from the currently occupied vertex to one of the other two vertices each with probability $\frac{1}{2}$). Find the probability that after n hops the flea is back where it started.

A second flea also hops about on the vertices of a triangle, but this flea is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops this second flea is back where it started?
[*Hint*: $\frac{1}{2} \pm \frac{i}{2\sqrt{3}} = \frac{1}{\sqrt{3}} e^{\pm i\pi/6}$.]

1.8 A die is 'fixed' so that when it is rolled the score cannot be the same as the previous score, all other scores having probability $\frac{1}{5}$. If the first score is 6, what is the probability p that the nth score is 6? What is the probability that the nth score is j, where $j \neq 6$?

Suppose instead that the die cannot score one greater (mod 6) than the previous score, all other five scores having equal probability. What is the new value of p?
[*Hint*: Think about the relationship between the two dice.]

1.9 Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{1,2,3\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ p & 1-p & 0 \end{pmatrix}.$$

Calculate $\mathbb{P}(X_n = 1 \mid X_0 = 1)$ in each of the following cases (a) $p = \frac{1}{16}$, (b) $p = \frac{1}{6}$, (c)* $p = \frac{1}{12}$.

1.10 Identify the communicating classes of the transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Which of the classes are closed?

1.11 Show that every transition matrix on a finite state-space has at least one closed communicating class. Find an example of a transition matrix with no closed communicating class.

1.12 A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if a player bets on the side which actually turns up, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10.

Let $X_0 = 2$ and let X_n $(n \geq 1)$ be his capital after n throws. Prove that the gambler will achieve his aim with probability $\frac{1}{5}$.

What is the expected number of tosses until the gambler either achieves his aim or loses his capital?

1.13 Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0,1,\dots\}$ with transition probabilities given by

$$p_{0,1} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1}, \quad i \geq 1.$$

Show that if $X_0 = 0$ the probability that $X_n \geq 1$ for all $n \geq 1$ is $6/\pi^2$.

1.14 Let $Y_1, Y_2, \dots$ be independent identically distributed random variables with $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$ and set $X_0 = 1$, $X_n = X_0 + Y_1 + \dots + Y_n$ for $n \geq 1$. Define

$$H_0 = \inf\{n \geq 0 : X_n = 0\}.$$

Find the probability generating function $\phi(s) = \mathbb{E}(s^{H_0})$.

Suppose the distribution of $Y_1, Y_2, \dots$ is changed to $\mathbb{P}(Y_1 = 2) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$. Show that ϕ now satisfies

$$s\phi^3 - 2\phi + s = 0.$$

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Example Sheet 2

2.1 Let X be a Markov chain containing an absorbing state s to which all other states lead (i.e., $j \to s$ for all j). Show that all states other than s are transient.

2.2 Let X be a Markov chain on $\{0, 1, 2, \dots\}$ with transition matrix given by $p_{0,j} = a_j$ for $j \geq 0$, $p_{i,i} = r$ and $p_{i,i-1} = 1 - r$ for $i \geq 1$. Assume that $0 < r < 1$. Classify the states of the chain, and find their mean recurrence times. [You may find it useful to define $J = \sup\{j : a_j > 0\}$.]

2.3 Compute $p_{11}(n)$ and classify the states of the Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$\begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{pmatrix}.$$

2.4 A particle performs a random walk on the vertices of a cube. At each step it remains where it is with probability $\frac{1}{4}$, and moves to each of its neighbouring vertices with probability $\frac{1}{4}$. Let v and w be two diametrically opposite vertices. If the walk starts at v, find
(a) the mean number of steps until its first return to v,
(b) the mean number of steps until its first visit to w,
(c) the mean number of visits to w before its first return to v.

2.5 In Exercise 1.10, which states are recurrent and which are transient?

2.6 What can be said about the number of visits to each state in the case where (a) a Markov chain is transient, and (b) a Markov chain is recurrent?

Consider the Markov chain $(X_n)_{n\geq 0}$ of Exercise 1.13. Show for this chain that $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty) = \mathbb{P}(\forall m,\ \exists n.\ X_N \geq m \text{ for all } N \geq n) = 1$.

Suppose the transition probabilities satisfy instead

$$p_{i,i+1} = \left(\frac{i+1}{i}\right)^{\alpha} p_{i,i-1}.$$

For each $\alpha \in (0, \infty)$ find the value of $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty)$.

2.7 The rooted binary tree is an infinite graph T with one distinguished vertex R from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on T jumps from a vertex along each available edge with equal probability. Show that the random walk is transient.

2.8 Show (by projection onto $\mathbb{Z}^3$ or otherwise) that the simple symmetric random walk in $\mathbb{Z}^4$ is transient.

2.9 Find all invariant distribution of the transition matrix in Exercise 1.10.

2.10 Two containers A and B are placed adjacently to one another, and gas is allowed to pass through a small aperture joining them. There are N molecules in all, and we assume that, at each epoch of time, one molecule (chosen at random) passes through the aperture. Show that the number of molecules in A evolves as a Markov chain. What are the transition probabilities? What is the invariant distribution of this chain? [This is the 'Ehrenfest urn model', first introduced by Ehrenfest under the name 'dog-flea model'.]

2.11 A fair die is thrown repeatedly. Let X_n denote the sum of the first n throws. Find

$$\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of } 13)$$

quoting carefully any general theorems that you use.

2.12 Find the invariant distributions of the transition matrices in Exercise 1.9, parts (a), (b) and (c), and compare them with your answers to that exercise.

2.13 Each morning a student takes one of the three books (labelled 1, 2, 3) he owns from his shelf. The probability that he chooses the book with label i is α_i (where $0 < \alpha_i < 1$, $i = 1, 2, 3$), and choices on successive days are independent. In the evening he replaces the book at the left-hand end of the shelf. If p_n denotes the probability that on day n the student finds the books in the order 1,2,3, from left to right, show that, irrespective of the initial arrangement of the books, p_n converges as $n \to \infty$, and determine the limit.

2.14 In each of the following cases determine whether the stochastic matrix P corresponds to a chain which is reversible in equilibrium:

(a) $\begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}$; (b) $\begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$;

(c) $I = \{0, 1, 2, \dots\}$ and $p_{01} = 1$, $p_{i,i+1} = p$, $p_{i,i-1} = 1-p$ for $i \geq 1$.
(d) $p_{ij} = p_{ji}$ for all $i, j \in S$.

2.15 A random walk on the set $\{0, 1, 2, \dots, b\}$ has transition matrix given by $p_{00} = 1 - \lambda_0$, $p_{bb} = 1 - \mu_b$, $p_{i,i+1} = \lambda_i$ and $p_{i+1,i} = \mu_{i+1}$ for $0 \leq i < b$, where $0 < \lambda_i, \mu_i < 1$ for all i, and $\lambda_i + \mu_i = 1$ for $1 \leq i < b$. Show that this process is time-reversible in equilibrium.

2.16 Let X be an irreducible non-null recurrent aperiodic Markov chain. Show that X is time-reversible in equilibrium if and only if

$$p_{j_1 j_2} p_{j_2 j_3} \cdots p_{j_{n-1} j_n} p_{j_n j_1} = p_{j_1 j_n} p_{j_n j_{n-1}} \cdots p_{j_2 j_1}$$

for all n and all finite sequences $j_1, j_2, \dots, j_n$ of states.