Linear Algebra: Example Sheet 1 of 4

1. Let $\mathbb{R}^{\mathbb{R}}$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication defined pointwise. Which of the following sets of functions form a vector subspace of $\mathbb{R}^{\mathbb{R}}$?

   (a) The set C of continuous functions.

   (b) The set $\{f \in C : |f(t)| \leq 1 \text{ for all } t \in [0,1]\}$.

   (c) The set $\{f \in C : f(t) \to 0 \text{ as } t \to \infty\}$.

   (d) The set $\{f \in C : f(t) \to 1 \text{ as } t \to \infty\}$.

   (e) The set of solutions of the differential equation $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = 0$.

   (f) The set of solutions of $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = \sin t$.

   (g) The set of solutions of $(\dot{x}(t))^2 - x(t) = 0$.

   (h) The set of solutions of $(\ddot{x}(t))^4 + (x(t))^2 = 0$.

2. Suppose that the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a basis for V. Which of the following are also bases?

   (a) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n$;

   (b) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n + \mathbf{e}_1$;

   (c) $\mathbf{e}_1 - \mathbf{e}_n, \mathbf{e}_2 + \mathbf{e}_{n-1}, \ldots, \mathbf{e}_n + (-1)^n \mathbf{e}_1$.

3. Let T, U and W be subspaces of V.
   (i) Show that $T \cup U$ is a subspace of V only if either $T \leq U$ or $U \leq T$.
   (ii) Give explicit counter-examples to the following statements:

   (a) $T + (U \cap W) = (T + U) \cap (T + W)$; (b) $(T + U) \cap W = (T \cap W) + (U \cap W)$.

   (iii) Show that each of the equalities in (ii) can be replaced by a valid inclusion of one side in the other.

4. For each of the following pairs of vector spaces (V, W) over $\mathbb{R}$, either give an isomorphism $V \to W$ or show that no such isomorphism can exist. *[Here P denotes the space of polynomial functions $\mathbb{R} \to \mathbb{R}$, and $C[a,b]$ denotes the space of continuous functions defined on the closed interval $[a,b]$.]*
   (a) $V = \mathbb{R}^4$, $W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$.
   (b) $V = \mathbb{R}^5$, $W = \{p \in P : \deg\ p \leq 5\}$.
   (c) $V = C[0,1]$, $W = C[-1,1]$.
   (d) $V = C[0,1]$, $W = \{f \in C[0,1] : f(0) = 0, f \text{ continuously differentiable }\}$.
   (e) $V = \mathbb{R}^2$, $W = \{\text{solutions of } \ddot{x}(t) + x(t) = 0\}$.
   (f) $V = \mathbb{R}^4$, $W = C[0,1]$.
   (g) (Harder:) $V = P$, $W = \mathbb{R}^{\mathbb{N}}$.

5. (i) If α and β are linear maps from U to V show that $\alpha + \beta$ is linear. Give explicit counter-examples to the following statements:

   (a) $\mathrm{Im}(\alpha + \beta) = \mathrm{Im}(\alpha) + \mathrm{Im}(\beta)$; (b) $\mathrm{Ker}(\alpha + \beta) = \mathrm{Ker}(\alpha) \cap \mathrm{Ker}(\beta)$.

   Show that in general each of these equalities can be replaced by a valid inclusion of one side in the other.
   (ii) Let α be a linear map from V to V. Show that if $\alpha^2 = \alpha$ then $V = \mathrm{Ker}(\alpha) \oplus \mathrm{Im}(\alpha)$. Does your proof still work if V is infinite dimensional? Is the result still true?

6. Let

$$U = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0,\ 2x_1 + 2x_2 + x_5 = 0\}, \quad W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0,\ x_2 = x_3 = x_4\}.$$

   Find bases for U and W containing a basis for $U \cap W$ as a subset. Give a basis for $U + W$ and show that

$$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}.$$

7. Let $\alpha: U \to V$ be a linear map between two finite dimensional vector spaces and let W be a vector subspace of U. Show that the restriction of α to W is a linear map $\alpha|_W: W \to V$ which satisfies

$$\mathrm{r}(\alpha) \geq \mathrm{r}(\alpha|_W) \geq \mathrm{r}(\alpha) - \dim(U) + \dim(W) \ .$$

Give examples (with $W \neq U$) to show that either of the two inequalities can be an equality.

8. (i) Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vector space V. Show that

$$V \geq \mathrm{Im}(\alpha) \geq \mathrm{Im}(\alpha^2) \geq \ldots \quad \text{and} \quad \{0\} \leq \mathrm{Ker}(\alpha) \leq \mathrm{Ker}(\alpha^2) \leq \ldots .$$

If $r_k = \mathrm{r}(\alpha^k)$, deduce that $r_k \geq r_{k+1}$ and that $r_k - r_{k+1} \geq r_{k+1} - r_{k+2}$. Conclude that if, for some $k \geq 0$, we have $r_k = r_{k+1}$, then $r_k = r_{k+\ell}$ for all $\ell \geq 0$.
(ii) Suppose that $\dim(V) = 5$, $\alpha^3 = 0$, but $\alpha^2 \neq 0$. What possibilities are there for $\mathrm{r}(\alpha)$ and $\mathrm{r}(\alpha^2)$?

9. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\alpha : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the matrix representing α relative to the basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for both the domain and the range.
Write down bases for the domain and range with respect to which the matrix of α is the identity.

10. Let $U_1, \ldots, U_k$ be subspaces of a vector space V and let B_i be a basis for U_i. Show that the following statements are equivalent:

    (i) $U = \sum_i U_i$ is a direct sum, *i.e.* every element of U can be written uniquely as $\sum_i u_i$ with $u_i \in U_i$.

    (ii) $U_j \cap \sum_{i \neq j} U_i = \{0\}$ for all j.

    (iii) The B_i are pairwise disjoint and their union is a basis for $\sum_i U_i$.

    Give an example where $U_i \cap U_j = \{0\}$ for all $i \neq j$, yet $U_1 + \ldots + U_k$ is not a direct sum.

11. Let Y and Z be subspaces of the finite dimensional vector spaces V and W, respectively. Show that $R = \{\alpha \in \mathcal{L}(V, W) : \alpha(Y) \leq Z\}$ is a subspace of the space $\mathcal{L}(V, W)$ of all linear maps from V to W. What is the dimension of R?

12. Recall that $\mathbb{F}^n$ has standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Let U be a subspace of $\mathbb{F}^n$. Show that there is a subset I of $\{1, 2, \ldots, n\}$ for which the subspace $W = \langle\{\mathbf{e}_i : i \in I\}\rangle$ is a complementary subspace to U in $\mathbb{F}^n$.

13. Suppose X and Y are linearly independent subsets of a vector space V; no member of X is expressible as a linear combination of members of Y, and no member of Y is expressible as a linear combination of members of X. Is the set $X \cup Y$ necessarily linearly independent? Give a proof or counterexample.

14. Show that any two subspaces of the same dimension in a finite dimensional real vector space have a common complementary subspace.

15. Let T, U, V, W be vector spaces over $\mathbb{F}$ and let $\alpha: T \to U$, $\beta: V \to W$ be fixed linear maps. Show that the mapping $\Phi: \mathcal{L}(U, V) \to \mathcal{L}(T, W)$ which sends θ to $\beta \circ \theta \circ \alpha$ is linear. If the spaces are finite-dimensional and α and β have rank r and s respectively, find the rank of Φ.

Linear Algebra: Example Sheet 2 of 4

1. Write down the three types of elementary matrices and find their inverses. Show that an $n \times n$ matrix A is invertible if and only if it can be written as a product of elementary matrices. Use this method to find the inverse of
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}.$$

2. (Another proof of the row rank column rank equality.) Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorises as $A = BC$ with $B \in \mathrm{Mat}_{m,r}(\mathbb{F})$ and $C \in \mathrm{Mat}_{r,n}(\mathbb{F})$. Using the fact that $(BC)^T = C^T B^T$, deduce that the (column) rank of A^T equals r.

3. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
(a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
(b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
(c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
(d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$.

4. Let P_n be the space of real polynomials of degree at most n. For $x \in \mathbb{R}$ define $\varepsilon_x \in P_n^*$ by $\varepsilon_x(p) = p(x)$. Show that $\varepsilon_0, \ldots, \varepsilon_n$ form a basis for P_n^*, and identify the basis of P_n to which it is dual.

5. (a) Show that if $\mathbf{x} \neq \mathbf{y}$ are vectors in the finite dimensional vector space V, then there is a linear functional $\theta \in V^*$ such that $\theta(\mathbf{x}) \neq \theta(\mathbf{y})$.
(b) Suppose that V is finite dimensional. Let $A, B \leq V$. Prove that $A \leq B$ if and only if $A^\circ \geq B^\circ$. Show that $A = V$ if and only if $A^\circ = \{\mathbf{0}\}$.

6. For $A \in \mathrm{Mat}_{n,m}(\mathbb{F})$ and $B \in \mathrm{Mat}_{m,n}(\mathbb{F})$, let $\tau_A(B)$ denote $\mathrm{tr} AB$. Show that, for each fixed A, $\tau_A \colon \mathrm{Mat}_{m,n}(\mathbb{F}) \to \mathbb{F}$ is linear. Show moreover that the mapping $A \mapsto \tau_A$ defines a linear isomorphism $\mathrm{Mat}_{n,m}(\mathbb{F}) \to \mathrm{Mat}_{m,n}(\mathbb{F})^*$.

7. (a) Let V be a non-zero finite dimensional real vector space. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

   (b) Let V be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Find endomorphisms α and β of V such that $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

8. Suppose that $\psi \colon U \times V \to \mathbb{F}$ is a bilinear form of rank r on finite dimensional vector spaces U and V over $\mathbb{F}$. Show that there exist bases $e_1, \ldots, e_m$ for U and $f_1, \ldots, f_n$ for V such that
$$\psi\left(\sum_{i=1}^{m} x_i e_i, \sum_{j=1}^{n} y_j f_j\right) = \sum_{k=1}^{r} x_k y_k$$
for all $x_1, \ldots, x_m, y_1, \ldots, y_n \in \mathbb{F}$. What are the dimensions of the left and right kernels of ψ?

9. Let A and B be $n \times n$ matrices over a field F. Show that the $2n \times 2n$ matrix
$$C = \begin{pmatrix} I & B \\ -A & 0 \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$
by elementary row operations (which you should specify). By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

10. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}\,(AB) = \mathrm{adj}\,(B)\mathrm{adj}\,(A)\,, \quad (ii)\ \det(\mathrm{adj}\,A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}\,(\mathrm{adj}\,A) = (\det A)^{n-2}A\,.$$

What happens if A is singular? *[Hint: Consider $A + \lambda I$ for $\lambda \in \mathbb{F}$.]*

Show that the rank of the adjugate matrix is $\mathrm{r}(\mathrm{adj}\,A) = \begin{cases} n & \text{if } \mathrm{r}(A) = n \\ 1 & \text{if } \mathrm{r}(A) = n-1 \\ 0 & \text{if } \mathrm{r}(A) \leq n-2. \end{cases}$

11. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \to \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.

In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:
(a) The map D defined by $D(p)(t) = p'(t)$.
(b) The map S defined by $S(p)(t) = p(t^2)$.
(c) The map E defined by $E(p)(t) = p(t-1)$.
(d) The composite DS.
(e) The composite SD.

Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

12. Suppose that $\psi: V \times V \to \mathbb{F}$ is a bilinear form on a finite dimensional vector space V. Take U a subspace of V with $U = W^{\perp}$ some subspace W of V. Suppose that $\psi|_{U \times U}$ is non-singular. Show that ψ is also non-singular.

13. Let V be a vector space. Suppose that $f_1, \ldots, f_n, g \in V^*$. Show that g is in the span of $f_1, \ldots, f_n$ if and only if $\bigcap_{i=1}^n \ker f_i \subset \ker g$.

14. Let $\alpha : V \to V$ be an endomorphism of a real finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
(i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
(ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.

The final question is based on non-examinable material

15. Let Y and Z be subspaces of the finite dimensional vector spaces V and W respectively. Suppose that $\alpha: V \to W$ is a linear map such that $\alpha(Y) \subset Z$. Show that α induces linear maps $\alpha|_Y: Y \to Z$ via $\alpha|_Y(y) = \alpha(y)$ and $\overline{\alpha}: V/Y \to W/Z$ via $\overline{\alpha}(v + Y) = \alpha(v) + Z$.

Consider a basis $(v_1, \ldots, v_n)$ for V containing a basis $(v_1, \ldots, v_k)$ for Y and a basis $(w_1, \ldots, w_m)$ for W containing a basis $(w_1, \ldots, w_l)$ for Z. Show that the matrix representing α with respect to $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ is a block matrix of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Explain how to determine the matrices representing $\alpha|_Y$ with respect to the bases $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_l)$ and representing $\overline{\alpha}$ with respect to the bases $(v_{k+1} + Y, \ldots, v_n + Y)$ and $(w_{l+1} + Z, \ldots, w_m + Z)$ from this block matrix.

Linear Algebra: Example Sheet 3 of 4

1. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

The second and third matrices commute; find a basis with respect to which they are both diagonal.

2. By considering the rank of a suitable matrix, find the eigenvalues of the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1. Hence write down the determinant of A.

3. Let α be an endomorphism of the finite dimensional vector space V over $\mathbb{F}$, with characteristic polynomial $\chi_\alpha(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_0$. Show that $\det(\alpha) = (-1)^n c_0$ and $\operatorname{tr}(\alpha) = -c_{n-1}$.

4. Let V be a vector space, let $\pi_1, \pi_2, \ldots, \pi_k$ be endomorphisms of V such that $\mathrm{id}_V = \pi_1 + \cdots + \pi_k$ and $\pi_i\pi_j = 0$ for any $i \neq j$. Show that $V = U_1 \oplus \cdots \oplus U_k$, where $U_j = \mathrm{Im}(\pi_j)$.
Let α be an endomorphism on the vector space V, satisfying the equation $\alpha^3 = \alpha$. Prove directly that $V = V_0 \oplus V_1 \oplus V_{-1}$, where V_λ is the λ-eigenspace of α.

5. Let α be an endomorphism of a finite dimensional complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. Are the eigenspaces $\mathrm{Ker}(\alpha - \lambda\iota)$ and $\mathrm{Ker}(\alpha^2 - \lambda^2\iota)$ necessarily the same?

6. (Another proof of the Diagonalisability Theorem.) Let V be a vector space of finite dimension. Show that if α_1 and α_2 are endomorphisms of V, then the nullity $n(\alpha_1\alpha_2)$ satisfies $n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2)$. Deduce that if α is an endomorphism of V such that $p(\alpha) = 0$ for some polynomial $p(t)$ which is a product of distinct linear factors, then α is diagonalisable.

7. Let A be a square complex matrix of finite order — that is, $A^m = I$ for some $m > 0$. Show that A can be diagonalised.

8. Show that none of the following matrices are similar:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Is the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

similar to any of them? If so, which?

9. Find a basis with respect to which $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ is in Jordan normal form. Hence compute $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^n$.

10. (a) Recall that the Jordan normal form of a 3×3 complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for 4×4 complex matrices.
(b) Let A be a 5×5 complex matrix with $A^4 = A^2 \neq A$. What are the possible minimal and characteristic polynomials? If A is not diagonalisable, how many possible JNFs are there for A?

11. Let V be a vector space of dimension n and α an endomorphism of V with $\alpha^n = 0$ but $\alpha^{n-1} \neq 0$. Show that there is a vector y such that $(y, \alpha(y), \alpha^2(y), \ldots, \alpha^{n-1}(y))$ is a basis for V.

Show that if β is an endomorphism of V which commutes with α, then $\beta = p(\alpha)$ for some polynomial p. *[Hint: consider $\beta(y)$.]* What is the form of the matrix for β with respect to the above basis?

12. Let α be an endomorphism of the finite-dimensional vector space V, and assume that α is invertible. Describe the eigenvalues and the characteristic and minimal polynomial of α^{-1} in terms of those of α.

13. Prove that that the inverse of a Jordan block $J_m(\lambda)$ with $\lambda \neq 0$ has Jordan normal form a Jordan block $J_m(\lambda^{-1})$. For an arbitrary invertible square matrix A, describe the Jordan normal form of A^{-1} in terms of that of A.

    Prove that any square complex matrix is similar to its transpose.

14. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are similar when regarded as matrices over $\mathbb{C}$, then they are similar as matrices over $\mathbb{R}$.

15. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_n & a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \ldots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

    Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

16. Let V denote the space of all infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$ and let α be the differentiation endomorphism $f \mapsto f'$.
    (i) Show that every real number λ is an eigenvalue of α. Show also that $\ker(\alpha - \lambda\iota)$ has dimension 1.
    (ii) Show that $\alpha - \lambda\iota$ is surjective for every real number λ.

Linear Algebra: Example Sheet 4 of 4

1. The square matrices A and B over the field F are congruent if $B = P^T AP$ for some invertible matrix P over F. Which of the following symmetric matrices are congruent to the identity matrix over $\mathbb{R}$, and which over $\mathbb{C}$? (Which, if any, over $\mathbb{Q}$?) Try to get away with the minimum calculation.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$

2. Find the rank and signature of the following quadratic forms over $\mathbb{R}$.

$$x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.$$

If A is the matrix of the first of these (say), find a non-singular matrix P such that $P^T AP$ is diagonal with entries ± 1.

3. (i) Show that the function $\psi(A, B) = \mathrm{tr}(AB^T)$ is a symmetric positive definite bilinear form on the space $\mathrm{Mat}_n(\mathbb{R})$ of all $n \times n$ real matrices. Deduce that $|\mathrm{tr}(AB^T)| \leq \mathrm{tr}(AA^T)^{1/2}\mathrm{tr}(BB^T)^{1/2}$.
(ii) Show that the map $A \mapsto \mathrm{tr}(A^2)$ is a quadratic form on $\mathrm{Mat}_n(\mathbb{R})$. Find its rank and signature.

4. Let $\psi : V \times V \to \mathbb{C}$ be a Hermitian form on a complex vector space V.
(i) Find the rank and signature of ψ in the case $V = \mathbb{C}^3$ and

$$\psi(x, x) = |x_1 + ix_2|^2 + |x_2 + ix_3|^2 + |x_3 + ix_1|^2 - |x_1 + x_2 + x_3|^2.$$

(ii) Show in general that if $n > 2$ then $\psi(u, v) = \frac{1}{n}\sum_{k=1}^{n} \zeta^{-k}\psi(u + \zeta^k v, u + \zeta^k v)$ where $\zeta = e^{2\pi i/n}$.

5. Show that the quadratic form $2(x^2+y^2+z^2+xy+yz+zx)$ is positive definite. *Write down* an orthonormal basis for the corresponding inner product on $\mathbb{R}^3$. Compute the basis of $\mathbb{R}^3$ obtained by applying the Gram-Schmidt process to the standard basis with respect to this inner product.

6. Let $W \leq V$ with V an inner product space. An endomorphism π of V is called an *idempotent* if $\pi^2 = \pi$. Show that the orthogonal projection onto W is a self-adjoint idempotent. Conversely show that any self-adjoint idempotent is orthogonal projection onto its image.

7. Let S be an $n \times n$ real symmetric matrix with $S^k = I$ for some $k \geq 1$. Show that $S^2 = I$.

8. An endomorphism α of a finite dimensional inner product space V is *positive definite* if it is self-adjoint and satisfies $\langle \alpha(\mathbf{x}), \mathbf{x} \rangle > 0$ for all non-zero $\mathbf{x} \in V$.
(i) Prove that a positive definite endomorphism has a unique positive definite square root.
(ii) Let α be an invertible endomorphism of V and α^* its adjoint. By considering $\alpha^*\alpha$, show that α can be factored as $\beta\gamma$ with β unitary and γ positive definite.

9. Let V be a finite dimensional complex inner product space, and let α be an endomorphism on V. Assume that α is *normal*, that is, α commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$. Show that α and α^* have a common eigenvector $\mathbf{v}$, and the corresponding eigenvalues are complex conjugates. Show that the subspace $\langle \mathbf{v} \rangle^\perp$ is invariant under both α and α^*. Deduce that there is an orthonormal basis of eigenvectors of α.

10. Find a linear transformation which simultaneously reduces the pair of real quadratic forms

$$2x^2 + 3y^2 + 3z^2 - 2yz, \qquad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6zx$$

to the forms

$$X^2 + Y^2 + Z^2, \qquad \lambda X^2 + \mu Y^2 + \nu Z^2$$

for some $\lambda, \mu, \nu \in \mathbb{R}$ (which should turn out in this example to be integers).

Does there exist a linear transformation which reduces the pair of real quadratic forms $x^2 - y^2$, $2xy$ simultaneously to diagonal forms?

11. Show that if A is an $m \times n$ real matrix of rank n then $A^T A$ is invertible. Find a corresponding result for complex matrices.

12. Let P_n be the ($n+1$-dimensional) space of real polynomials of degree $\leq n$. Define

$$(f, g) = \int_{-1}^{+1} f(t)g(t)dt\,.$$

Show that $(\ ,\ )$ is an inner product on P_n and that the endomorphism $\alpha : P_n \to P_n$ defined by

$$\alpha(f)(t) = (1 - t^2)f''(t) - 2tf'(t)$$

is self-adjoint. What are the eigenvalues of α?

Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1 - t^2)^k$. Prove the following.
(i) For $i \neq j$, $(s_i, s_j) = 0$.
(ii) $s_0, \ldots, s_n$ forms a basis for P_n.
(iii) For all $1 \leq k \leq n$, s_k spans the orthogonal complement of P_{k-1} in P_k.
(iv) s_k is an eigenvector of α. (Give its eigenvalue.)
What is the relation between the s_k and the result of applying Gram-Schmidt to the sequence 1, x, x^2, x^3 and so on? (Calculate the first few terms?)

13. Let $f_1, \cdots, f_t, f_{t+1}, \cdots, f_{t+u}$ be linear functionals on the finite dimensional real vector space V. Show that $Q(\mathbf{x}) = f_1(\mathbf{x})^2 + \cdots + f_t(\mathbf{x})^2 - f_{t+1}(\mathbf{x})^2 - \cdots - f_{t+u}(\mathbf{x})^2$ is a quadratic form on V. Suppose Q has rank $p + q$ and signature $p - q$. Show that $p \leq t$ and $q \leq u$.

14. Let a_1, a_2, ... , a_n be real numbers such that $a_1 + \cdots + a_n = 0$ and $a_1^2 + \cdots + a_n^2 = 1$. What is the maximum value of $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$?

15. Suppose that α is an orthogonal endomorphism on the finite-dimensional real inner product space V. Prove that V can be decomposed into a direct sum of mutually orthogonal α-invariant subspaces of dimension 1 or 2. Determine the possible matrices of α with respect to orthonormal bases in the cases where V has dimension 1 or dimension 2.