

# Part IB — Linear Algebra

## Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Definition of a vector space (over  $\mathbb{R}$  or  $\mathbb{C}$ ), subspaces, the space spanned by a subset. Linear independence, bases, dimension. Direct sums and complementary subspaces. [3]

Linear maps, isomorphisms. Relation between rank and nullity. The space of linear maps from  $U$  to  $V$ , representation by matrices. Change of basis. Row rank and column rank. [4]

Determinant and trace of a square matrix. Determinant of a product of two matrices and of the inverse matrix. Determinant of an endomorphism. The adjugate matrix. [3]

Eigenvalues and eigenvectors. Diagonal and triangular forms. Characteristic and minimal polynomials. Cayley-Hamilton Theorem over  $\mathbb{C}$ . Algebraic and geometric multiplicity of eigenvalues. Statement and illustration of Jordan normal form. [4]

Dual of a finite-dimensional vector space, dual bases and maps. Matrix representation, rank and determinant of dual map. [2]

Bilinear forms. Matrix representation, change of basis. Symmetric forms and their link with quadratic forms. Diagonalisation of quadratic forms. Law of inertia, classification by rank and signature. Complex Hermitian forms. [4]

Inner product spaces, orthonormal sets, orthogonal projection,  $V = W \oplus W^\perp$ . Gram-Schmidt orthogonalisation. Adjoints. Diagonalisation of Hermitian matrices. Orthogonality of eigenvectors and properties of eigenvalues. [4]

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## 0 Introduction

# 1 Vector spaces

## 1.1 Definitions and examples

**Notation.** We will use  $\mathbb{F}$  to denote an arbitrary field, usually  $\mathbb{R}$  or  $\mathbb{C}$ .

**Definition** (Vector space). An  $\mathbb{F}$ -vector space is an (additive) abelian group  $V$  together with a function  $\mathbb{F} \times V \rightarrow V$ , written  $(\lambda, \mathbf{v}) \mapsto \lambda\mathbf{v}$ , such that

- (i)  $\lambda(\mu\mathbf{v}) = \lambda\mu\mathbf{v}$  for all  $\lambda, \mu \in \mathbb{F}$ ,  $\mathbf{v} \in V$  (associativity)
- (ii)  $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$  for all  $\lambda \in \mathbb{F}$ ,  $\mathbf{u}, \mathbf{v} \in V$  (distributivity in  $V$ )
- (iii)  $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$  for all  $\lambda, \mu \in \mathbb{F}$ ,  $\mathbf{v} \in V$  (distributivity in  $\mathbb{F}$ )
- (iv)  $1\mathbf{v} = \mathbf{v}$  for all  $\mathbf{v} \in V$  (identity)

We always write  $\mathbf{0}$  for the additive identity in  $V$ , and call this the identity. By abuse of notation, we also write  $0$  for the trivial vector space  $\{\mathbf{0}\}$ .

**Definition** (Subspace). If  $V$  is an  $\mathbb{F}$ -vector space, then  $U \subseteq V$  is an ( $\mathbb{F}$ -linear) subspace if

- (i)  $\mathbf{u}, \mathbf{v} \in U$  implies  $\mathbf{u} + \mathbf{v} \in U$ .
- (ii)  $\mathbf{u} \in U, \lambda \in \mathbb{F}$  implies  $\lambda\mathbf{u} \in U$ .
- (iii)  $\mathbf{0} \in U$ .

These conditions can be expressed more concisely as “ $U$  is non-empty and if  $\lambda, \mu \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in U$ , then  $\lambda\mathbf{u} + \mu\mathbf{v} \in U$ ”.

Alternatively,  $U$  is a subspace of  $V$  if it is itself a vector space, inheriting the operations from  $V$ .

We sometimes write  $U \leq V$  if  $U$  is a subspace of  $V$ .

**Definition** (Sum of subspaces). Suppose  $U, W$  are subspaces of an  $\mathbb{F}$  vector space  $V$ . The *sum* of  $U$  and  $W$  is

$$U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}.$$

**Definition** (Quotient spaces). Let  $V$  be a vector space, and  $U \subseteq V$  a subspace. Then the quotient group  $V/U$  can be made into a vector space called the *quotient space*, where scalar multiplication is given by  $(\lambda, \mathbf{v} + U) = (\lambda\mathbf{v}) + U$ .

This is well defined since if  $\mathbf{v} + U = \mathbf{w} + U \in V/U$ , then  $\mathbf{v} - \mathbf{w} \in U$ . Hence for  $\lambda \in \mathbb{F}$ , we have  $\lambda\mathbf{v} - \lambda\mathbf{w} \in U$ . So  $\lambda\mathbf{v} + U = \lambda\mathbf{w} + U$ .

## 1.2 Linear independence, bases and the Steinitz exchange lemma

**Definition** (Span). Let  $V$  be a vector space over  $\mathbb{F}$  and  $S \subseteq V$ . The *span* of  $S$  is defined as

$$\langle S \rangle = \left\{ \sum_{i=1}^n \lambda_i \mathbf{s}_i : \lambda_i \in \mathbb{F}, \mathbf{s}_i \in S, n \geq 0 \right\}$$

This is the smallest subspace of  $V$  containing  $S$ .

Note that the sums must be finite. We will not play with infinite sums, since the notion of convergence is not even well defined in a general vector space.

**Definition** (Spanning set). Let  $V$  be a vector space over  $\mathbb{F}$  and  $S \subseteq V$ .  $S$  spans  $V$  if  $\langle S \rangle = V$ .

**Definition** (Linear independence). Let  $V$  be a vector space over  $\mathbb{F}$  and  $S \subseteq V$ . Then  $S$  is *linearly independent* if whenever

$$\sum_{i=1}^n \lambda_i \mathbf{s}_i = \mathbf{0} \text{ with } \lambda_i \in \mathbb{F}, \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n \in S \text{ distinct,}$$

we must have  $\lambda_i = 0$  for all  $i$ .

If  $S$  is not linearly independent, we say it is *linearly dependent*.

**Definition** (Basis). Let  $V$  be a vector space over  $\mathbb{F}$  and  $S \subseteq V$ . Then  $S$  is a *basis* for  $V$  if  $S$  is linearly independent and spans  $V$ .

**Definition** (Finite dimensional). A vector space is *finite dimensional* if there is a finite basis.

**Definition** (Dimension). If  $V$  is a vector space over  $\mathbb{F}$  with finite basis  $S$ , then the *dimension* of  $V$ , written

$$\dim V = \dim_{\mathbb{F}} V = |S|.$$

### 1.3 Direct sums

**Definition** ((Internal) direct sum). Suppose  $V$  is a vector space over  $\mathbb{F}$  and  $U, W \subseteq V$  are subspaces. We say that  $V$  is the (*internal*) *direct sum* of  $U$  and  $W$  if

$$(i) \quad U + W = V$$

$$(ii) \quad U \cap W = \mathbf{0}.$$

We write  $V = U \oplus W$ .

Equivalently, this requires that every  $\mathbf{v} \in V$  can be written uniquely as  $\mathbf{u} + \mathbf{w}$  with  $\mathbf{u} \in U, \mathbf{w} \in W$ . We say that  $U$  and  $W$  are *complementary subspaces* of  $V$ .

**Definition** ((External) direct sum). If  $U, W$  are vector spaces over  $\mathbb{F}$ , the (*external*) *direct sum* is

$$U \oplus W = \{(\mathbf{u}, \mathbf{w}) : \mathbf{u} \in U, \mathbf{w} \in W\},$$

with addition and scalar multiplication componentwise:

$$(\mathbf{u}_1, \mathbf{w}_1) + (\mathbf{u}_2, \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{w}_1 + \mathbf{w}_2), \quad \lambda(\mathbf{u}, \mathbf{w}) = (\lambda\mathbf{u}, \lambda\mathbf{w}).$$

**Definition** ((Multiple) (internal) direct sum). If  $U_1, \dots, U_n \subseteq V$  are subspaces of  $V$ , then  $V$  is the (*internal*) *direct sum*

$$V = U_1 \oplus \dots \oplus U_n = \bigoplus_{i=1}^n U_i$$

if every  $\mathbf{v} \in V$  can be written uniquely as  $\mathbf{v} = \sum \mathbf{u}_i$  with  $\mathbf{u}_i \in U_i$ .

This can be extended to an infinite sum with the same definition, just noting that the sum  $\mathbf{v} = \sum \mathbf{u}_i$  has to be finite.

**Definition** ((Multiple) (external) direct sum). If  $U_1, \dots, U_n$  are vector spaces over  $\mathbb{F}$ , the external direct sum is

$$U_1 \oplus \dots \oplus U_n = \bigoplus_{i=1}^n U_i = \{(\mathbf{u}_1, \dots, \mathbf{u}_n) : \mathbf{u}_i \in U_i\},$$

with pointwise operations.

This can be made into an infinite sum if we require that all but finitely many of the  $\mathbf{u}_i$  have to be zero.

## 2 Linear maps

### 2.1 Definitions and examples

**Definition** (Linear map). Let  $U, V$  be vector spaces over  $\mathbb{F}$ . Then  $\alpha : U \rightarrow V$  is a *linear map* if

- (i)  $\alpha(\mathbf{u}_1 + \mathbf{u}_2) = \alpha(\mathbf{u}_1) + \alpha(\mathbf{u}_2)$  for all  $\mathbf{u}_i \in U$ .
- (ii)  $\alpha(\lambda\mathbf{u}) = \lambda\alpha(\mathbf{u})$  for all  $\lambda \in \mathbb{F}, \mathbf{u} \in U$ .

We write  $\mathcal{L}(U, V)$  for the set of linear maps  $U \rightarrow V$ .

**Definition** (Isomorphism). We say a linear map  $\alpha : U \rightarrow V$  is an *isomorphism* if there is some  $\beta : V \rightarrow U$  (also linear) such that  $\alpha \circ \beta = \text{id}_V$  and  $\beta \circ \alpha = \text{id}_U$ .

If there exists an isomorphism  $U \rightarrow V$ , we say  $U$  and  $V$  are *isomorphic*, and write  $U \cong V$ .

**Definition** (Image and kernel). Let  $\alpha : U \rightarrow V$  be a linear map. Then the *image* of  $\alpha$  is

$$\text{im } \alpha = \{\alpha(\mathbf{u}) : \mathbf{u} \in U\}.$$

The *kernel* of  $\alpha$  is

$$\text{ker } \alpha = \{\mathbf{u} : \alpha(\mathbf{u}) = \mathbf{0}\}.$$

### 2.2 Linear maps and matrices

**Definition** (Matrix representation). We call the matrix corresponding to a linear map  $\alpha \in \mathcal{L}(U, V)$  under the corollary the *matrix representing*  $\alpha$  with respect to the bases  $(\mathbf{e}_1, \dots, \mathbf{e}_m)$  and  $(\mathbf{f}_1, \dots, \mathbf{f}_n)$ .

### 2.3 The first isomorphism theorem and the rank-nullity theorem

**Definition** (Rank and nullity). If  $\alpha : U \rightarrow V$  is a linear map between finite-dimensional vector spaces over  $\mathbb{F}$  (in fact we just need  $U$  to be finite-dimensional), the *rank* of  $\alpha$  is the number  $r(\alpha) = \dim \text{im } \alpha$ . The *nullity* of  $\alpha$  is the number  $n(\alpha) = \dim \text{ker } \alpha$ .

### 2.4 Change of basis

**Definition** (Equivalent matrices). We say  $A, B \in \text{Mat}_{n,m}(\mathbb{F})$  are *equivalent* if there are invertible matrices  $P \in \text{Mat}_m(\mathbb{F}), Q \in \text{Mat}_n(\mathbb{F})$  such that  $B = Q^{-1}AP$ .

**Definition** (Column and row rank). If  $A \in \text{Mat}_{n,m}(\mathbb{F})$ , then

- The *column rank* of  $A$ , written  $r(A)$ , is the dimension of the subspace of  $\mathbb{F}^n$  spanned by the columns of  $A$ .
- The *row rank* of  $A$ , written  $r(A)$ , is the dimension of the subspace of  $\mathbb{F}^m$  spanned by the rows of  $A$ . Alternatively, it is the column rank of  $A^T$ .

## 2.5 Elementary matrix operations

**Definition** (Elementary matrices). We call the following matrices of  $GL_n(\mathbb{F})$  *elementary matrices*:

$$S_{ij}^n = \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 0 & & & 1 & & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & 1 & & & & & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix}$$

This is called a reflection, where the rows we changed are the  $i$ th and  $j$ th row.

$$E_{ij}^n(\lambda) = \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \lambda & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & 1 & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & 1 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & 1 & \\ & & & & & & & & & & 1 \end{pmatrix}$$

This is called a shear, where  $\lambda$  appears at the  $i, j$ th entry.

$$T_i^n(\lambda) = \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \lambda & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & 1 & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & 1 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & 1 & \\ & & & & & & & & & & 1 \end{pmatrix}$$

This is called a shear, where  $\lambda \neq 0$  appears at the  $i$ th column.



## 3 Duality

### 3.1 Dual space

**Definition** (Dual space). Let  $V$  be a vector space over  $\mathbb{F}$ . The *dual* of  $V$  is defined as

$$V^* = \mathcal{L}(V, \mathbb{F}) = \{\theta : V \rightarrow \mathbb{F} : \theta \text{ linear}\}.$$

Elements of  $V^*$  are called *linear functionals* or *linear forms*.

**Definition** (Annihilator). Let  $U \subseteq V$ . Then the *annihilator* of  $U$  is

$$U^0 = \{\theta \in V^* : \theta(\mathbf{u}) = 0, \forall \mathbf{u} \in U\}.$$

If  $W \subseteq V^*$ , then the *annihilator* of  $W$  is

$$W^0 = \{\mathbf{v} \in V : \theta(\mathbf{v}) = 0, \forall \theta \in W\}.$$

### 3.2 Dual maps

**Definition** (Dual map). Let  $V, W$  be vector spaces over  $\mathbb{F}$  and  $\alpha : V \rightarrow W \in \mathcal{L}(V, W)$ . The *dual map* to  $\alpha$ , written  $\alpha^* : W^* \rightarrow V^*$  is given by  $\theta \mapsto \theta \circ \alpha$ . Since the composite of linear maps is linear,  $\alpha^*(\theta) \in V^*$ . So this is a genuine map.

## 4 Bilinear forms I

**Definition** (Bilinear form). Let  $V, W$  be vector spaces over  $\mathbb{F}$ . Then a function  $\phi : V \times W \rightarrow \mathbb{F}$  is a *bilinear form* if it is linear in each variable, i.e. for each  $\mathbf{v} \in V$ ,  $\phi(\mathbf{v}, \cdot) : W \rightarrow \mathbb{F}$  is linear; for each  $\mathbf{w} \in W$ ,  $\phi(\cdot, \mathbf{w}) : V \rightarrow \mathbb{F}$  is linear.

**Definition** (Matrix representing bilinear form). Let  $(\mathbf{e}_1, \dots, \mathbf{e}_n)$  be a basis for  $V$  and  $(\mathbf{f}_1, \dots, \mathbf{f}_m)$  be a basis for  $W$ , and  $\psi : V \times W \rightarrow \mathbb{F}$ . Then the *matrix*  $A$  representing  $\psi$  with respect to the basis is defined to be

$$A_{ij} = \psi(\mathbf{e}_i, \mathbf{f}_j).$$

**Definition** (Left and right kernel). The kernel of  $\psi_L$  is *left kernel* of  $\psi$ , while the kernel of  $\psi_R$  is the *right kernel* of  $\psi$ .

**Definition** (Non-degenerate bilinear form).  $\psi$  is *non-degenerate* if the left and right kernels are both trivial. We say  $\psi$  is *degenerate* otherwise.

**Definition** (Rank of bilinear form). If  $\psi : V \times W$  is a bilinear form on a finite-dimensional vector space  $V$ , then the *rank* of  $V$  is the rank of any matrix representing  $\phi$ . This is well-defined since  $r(P^T A Q) = r(A)$  if  $P$  and  $Q$  are invertible.

Alternatively, it is the rank of  $\psi_L$  (or  $\psi_R$ ).

## 5 Determinants of matrices

**Definition** (Determinant). Let  $A \in \text{Mat}_{n,n}(\mathbb{F})$ . Its *determinant* is

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^n A_{i\sigma(i)}.$$

**Definition** (Volume form). A *volume form* on  $\mathbb{F}^n$  is a function  $d : \mathbb{F}^n \times \cdots \times \mathbb{F}^n \rightarrow \mathbb{F}$  that is

- (i) Multilinear, i.e. for all  $i$  and all  $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n \in \mathbb{F}^n$ , we have

$$d(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \cdot, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n) \in (\mathbb{F}^n)^*.$$

- (ii) Alternating, i.e. if  $\mathbf{v}_i = \mathbf{v}_j$  for some  $i \neq j$ , then

$$d(\mathbf{v}_1, \dots, \mathbf{v}_n) = 0.$$

**Definition** (Singular matrices). A matrix  $A$  is *singular* if  $\det A = 0$ . Otherwise, it is *non-singular*.

**Notation.** Write  $\hat{A}_{ij}$  for the matrix obtained from  $A$  by deleting the  $i$ th row and  $j$ th column.

**Definition** (Adjugate matrix). Let  $A \in \text{Mat}_n(\mathbb{F})$ . The *adjugate matrix* of  $A$ , written  $\text{adj } A$ , is the  $n \times n$  matrix such that  $(\text{adj } A)_{ij} = (-1)^{i+j} \det \hat{A}_{ji}$ .

## 6 Endomorphisms

### 6.1 Invariants

**Definition.** If  $V$  is a (finite-dimensional) vector space over  $\mathbb{F}$ . An *endomorphism* of  $V$  is a linear map  $\alpha : V \rightarrow V$ . We write  $\text{End}(V)$  for the  $\mathbb{F}$ -vector space of all such linear maps, and  $I$  for the identity map  $V \rightarrow V$ .

**Definition** (Similar matrices). We say matrices  $A$  and  $B$  are *similar* or *conjugate* if there is some  $P$  invertible such that  $B = P^{-1}AP$ .

**Definition** (Trace). The *trace* of a matrix of  $A \in \text{Mat}_n(\mathbb{F})$  is defined by

$$\text{tr } A = \sum_{i=1}^n A_{ii}.$$

**Definition** (Trace and determinant of endomorphism). Let  $\alpha \in \text{End}(V)$ , and  $A$  be a matrix representing  $\alpha$  under any basis. Then the *trace* of  $\alpha$  is  $\text{tr } \alpha = \text{tr } A$ , and the *determinant* is  $\det \alpha = \det A$ .

**Definition** (Eigenvalue and eigenvector). Let  $\alpha \in \text{End}(V)$ . Then  $\lambda \in \mathbb{F}$  is an *eigenvalue* (or *E-value*) if there is some  $\mathbf{v} \in V \setminus \{0\}$  such that  $\alpha\mathbf{v} = \lambda\mathbf{v}$ .

$\mathbf{v}$  is an *eigenvector* if  $\alpha(\mathbf{v}) = \lambda\mathbf{v}$  for some  $\lambda \in \mathbb{F}$ .

When  $\lambda \in \mathbb{F}$ , the  $\lambda$ -*eigenspace*, written  $E_\alpha(\lambda)$  or  $E(\lambda)$  is the subspace of  $V$  containing all the  $\lambda$ -eigenvectors, i.e.

$$E_\alpha(\lambda) = \ker(\lambda\iota - \alpha).$$

where  $\iota$  is the identity function.

**Definition** (Characteristic polynomial). The *characteristic polynomial* of  $\alpha$  is defined by

$$\chi_\alpha(t) = \det(t\iota - \alpha).$$

**Definition** (Diagonalizable). We say  $\alpha \in \text{End}(V)$  is diagonalizable if there is some basis for  $V$  such that  $\alpha$  is represented by a diagonal matrix, i.e. all terms not on the diagonal are zero.

### 6.2 The minimal polynomial

#### 6.2.1 Aside on polynomials

**Definition** (Polynomial). A *polynomial* over  $\mathbb{F}$  is an object of the form

$$f(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_1 t + a_0,$$

with  $m \geq 0, a_0, \dots, a_m \in \mathbb{F}$ .

We write  $\mathbb{F}[t]$  for the set of polynomials over  $\mathbb{F}$ .

**Definition** (Degree). Let  $f \in \mathbb{F}[t]$ . Then the *degree* of  $f$ , written  $\deg f$  is the largest  $n$  such that  $a_n \neq 0$ . In particular,  $\deg 0 = -\infty$ .

**Definition** (Multiplicity of a root). Let  $f \in \mathbb{F}[t]$  and  $\lambda$  a root of  $f$ . We say  $\lambda$  has *multiplicity*  $k$  if  $(t - \lambda)^k$  is a factor of  $f$  but  $(t - \lambda)^{k+1}$  is not, i.e.

$$f(t) = (t - \lambda)^k g(t)$$

for some  $g(t) \in \mathbb{F}[t]$  with  $g(\lambda) \neq 0$ .

### 6.2.2 Minimal polynomial

**Notation.** Given  $f(t) = \sum_{i=0}^m a_i t^i \in \mathbb{F}[t]$ ,  $A \in \text{Mat}_n(\mathbb{F})$  and  $\alpha \in \text{End}(V)$ , we can write

$$f(A) = \sum_{i=0}^m a_i A^i, \quad f(\alpha) = \sum_{i=0}^m a_i \alpha^i$$

where  $A^0 = I$  and  $\alpha^0 = \iota$ .

**Definition** (Minimal polynomial). The *minimal polynomial* of  $\alpha \in \text{End}(V)$  is the non-zero monic polynomial  $M_\alpha(t)$  of least degree such that  $M_\alpha(\alpha) = 0$ .

### 6.3 The Cayley-Hamilton theorem

**Definition** (Triangulable). An endomorphism  $\alpha \in \text{End}(V)$  is *triangulable* if there is a basis for  $V$  such that  $\alpha$  is represented by an upper triangular matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

### 6.4 Multiplicities of eigenvalues and Jordan normal form

**Definition** (Algebraic and geometry multiplicity). Let  $\alpha \in \text{End}(V)$  and  $\lambda$  an eigenvalue of  $\alpha$ . Then

- (i) The *algebraic multiplicity* of  $\lambda$ , written  $a_\lambda$ , is the multiplicity of  $\lambda$  as a root of  $\chi_\alpha(t)$ .
- (ii) The *geometric multiplicity* of  $\lambda$ , written  $g_\lambda$ , is the dimension of the corresponding eigenspace,  $\dim E_\alpha(\lambda)$ .
- (iii)  $c_\lambda$  is the multiplicity of  $\lambda$  as a root of the minimal polynomial  $m_\alpha(t)$ .

**Definition** (Jordan normal form). We say  $A \in \text{Mat}_N(\mathbb{C})$  is in *Jordan normal form* if it is a block diagonal of the form

$$\begin{pmatrix} J_{n_1}(\lambda_1) & & & 0 \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{n_k}(\lambda_k) \end{pmatrix}$$

where  $k \geq 1$ ,  $n_1, \dots, n_k \in \mathbb{N}$  such that  $n = \sum n_i$ ,  $\lambda_1, \dots, \lambda_k$  not necessarily distinct, and

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$$

is an  $m \times m$  matrix. Note that  $J_m(\lambda) = \lambda I_m + J_m(0)$ .

**Definition** (Nilpotent). We say  $\alpha \in \text{End}(V)$  is nilpotent if there is some  $r$  such that  $\alpha^r = 0$ .

## 7 Bilinear forms II

### 7.1 Symmetric bilinear forms and quadratic forms

**Definition** (Symmetric bilinear form). Let  $V$  is a vector space over  $\mathbb{F}$ . A bilinear form  $\phi : V \times V \rightarrow \mathbb{F}$  is *symmetric* if

$$\phi(\mathbf{v}, \mathbf{w}) = \phi(\mathbf{w}, \mathbf{v})$$

for all  $\mathbf{v}, \mathbf{w} \in V$ .

**Definition** (Congruent matrices). Two square matrices  $A, B$  are *congruent* if there exists some invertible  $P$  such that

$$B = P^T A P.$$

**Definition** (Quadratic form). A function  $q : V \rightarrow \mathbb{F}$  is a *quadratic form* if there exists some bilinear form  $\phi$  such that

$$q(\mathbf{v}) = \phi(\mathbf{v}, \mathbf{v})$$

for all  $\mathbf{v} \in V$ .

**Definition** (Positive/negative (semi-)definite). Let  $\phi$  be a symmetric bilinear form on a finite-dimensional real vector space  $V$ . We say

- (i)  $\phi$  is *positive definite* if  $\phi(\mathbf{v}, \mathbf{v}) > 0$  for all  $\mathbf{v} \in V \setminus \{0\}$ .
- (ii)  $\phi$  is *positive semi-definite* if  $\phi(\mathbf{v}, \mathbf{v}) \geq 0$  for all  $\mathbf{v} \in V$ .
- (iii)  $\phi$  is *negative definite* if  $\phi(\mathbf{v}, \mathbf{v}) < 0$  for all  $\mathbf{v} \in V \setminus \{0\}$ .
- (iv)  $\phi$  is *negative semi-definite* if  $\phi(\mathbf{v}, \mathbf{v}) \leq 0$  for all  $\mathbf{v} \in V$ .

**Definition** (Signature). The *signature* of a bilinear form  $\phi$  is the number  $p - q$ , where  $p$  and  $q$  are as above.

### 7.2 Hermitian form

**Definition** (Sesquilinear form). Let  $V, W$  be complex vector spaces. Then a *sesquilinear form* is a function  $\phi : V \times W \rightarrow \mathbb{C}$  such that

- (i)  $\phi(\lambda \mathbf{v}_1 + \mu \mathbf{v}_2, \mathbf{w}) = \bar{\lambda} \phi(\mathbf{v}_1, \mathbf{w}) + \bar{\mu} \phi(\mathbf{v}_2, \mathbf{w})$ .
- (ii)  $\phi(\mathbf{v}, \lambda \mathbf{w}_1 + \mu \mathbf{w}_2) = \lambda \phi(\mathbf{v}, \mathbf{w}_1) + \mu \phi(\mathbf{v}, \mathbf{w}_2)$ .

for all  $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V, \mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in W$  and  $\lambda, \mu \in \mathbb{C}$ .

**Definition** (Representation of sesquilinear form). Let  $V, W$  be finite-dimensional complex vector spaces with basis  $(\mathbf{v}_1, \dots, \mathbf{v}_n)$  and  $(\mathbf{w}_1, \dots, \mathbf{w}_m)$  respectively, and  $\phi : V \times W \rightarrow \mathbb{C}$  be a sesquilinear form. Then the matrix representing  $\phi$  with respect to these bases is

$$A_{ij} = \phi(\mathbf{v}_i, \mathbf{w}_j).$$

for  $1 \leq i \leq n, 1 \leq j \leq m$ .

**Definition** (Hermitian sesquilinear form). A sesquilinear form on  $V \times V$  is *Hermitian* if

$$\phi(\mathbf{v}, \mathbf{w}) = \overline{\phi(\mathbf{w}, \mathbf{v})}.$$

## 8 Inner product spaces

### 8.1 Definitions and basic properties

**Definition** (Inner product space). Let  $V$  be a vector space. An *inner product* on  $V$  is a positive-definite symmetric bilinear/hermitian form. We usually write  $(x, y)$  instead of  $\phi(x, y)$ .

A vector space equipped with an inner product is an *inner product space*.

**Definition** (Orthogonal vectors). Let  $V$  be an inner product space. Then  $\mathbf{v}, \mathbf{w} \in V$  are *orthogonal* if  $(\mathbf{v}, \mathbf{w}) = 0$ .

**Definition** (Orthonormal set). Let  $V$  be an inner product space. A set  $\{\mathbf{v}_i : i \in I\}$  is an *orthonormal set* if for any  $i, j \in I$ , we have

$$(\mathbf{v}_i, \mathbf{v}_j) = \delta_{ij}$$

**Definition** (Orthonormal basis). Let  $V$  be an inner product space. A subset of  $V$  is an *orthonormal basis* if it is an orthonormal set and is a basis.

### 8.2 Gram-Schmidt orthogonalization

**Definition** (Orthogonal internal direct sum). Let  $V$  be an inner product space and  $V_1, V_2 \leq V$ . Then  $V$  is the *orthogonal internal direct sum* of  $V_1$  and  $V_2$  if it is a direct sum and  $V_1$  and  $V_2$  are orthogonal. More precisely, we require

- (i)  $V = V_1 + V_2$
- (ii)  $V_1 \cap V_2 = 0$
- (iii)  $(\mathbf{v}_1, \mathbf{v}_2) = 0$  for all  $\mathbf{v}_1 \in V_1$  and  $\mathbf{v}_2 \in V_2$ .

Note that condition (iii) implies (ii), but we write it for the sake of explicitness.

We write  $V = V_1 \perp V_2$ .

**Definition** (Orthogonal complement). If  $W \leq V$  is a subspace of an inner product space  $V$ , then the *orthogonal complement* of  $W$  in  $V$  is the subspace

$$W^\perp = \{\mathbf{v} \in V : (\mathbf{v}, \mathbf{w}) = 0, \forall \mathbf{w} \in W\}.$$

**Definition** (Orthogonal external direct sum). Let  $V_1, V_2$  be inner product spaces. The *orthogonal external direct sum* of  $V_1$  and  $V_2$  is the vector space  $V_1 \oplus V_2$  with the inner product defined by

$$(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2) = (\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2),$$

with  $\mathbf{v}_1, \mathbf{w}_1 \in V_1, \mathbf{v}_2, \mathbf{w}_2 \in V_2$ .

Here we write  $\mathbf{v}_1 + \mathbf{v}_2 \in V_1 \oplus V_2$  instead of  $(\mathbf{v}_1, \mathbf{v}_2)$  to avoid confusion.

### 8.3 Adjoints, orthogonal and unitary maps

**Definition** (Adjoint). We call the map  $\alpha^*$  the *adjoint* of  $\alpha$ .

**Definition** (Self-adjoint). Let  $V$  be an inner product space, and  $\alpha \in \text{End}(V)$ . Then  $\alpha$  is *self-adjoint* if  $\alpha = \alpha^*$ , i.e.

$$(\alpha(\mathbf{v}), \mathbf{w}) = (\mathbf{v}, \alpha(\mathbf{w}))$$

for all  $\mathbf{v}, \mathbf{w}$ .

**Definition** (Orthogonal endomorphism). Let  $V$  be a real inner product space. Then  $\alpha \in \text{End}(V)$  is *orthogonal* if

$$(\alpha(\mathbf{v}), \alpha(\mathbf{w})) = (\mathbf{v}, \mathbf{w})$$

for all  $\mathbf{v}, \mathbf{w} \in V$ .

**Definition** (Orthogonal group). Let  $V$  be a real inner product space. Then the *orthogonal group* of  $V$  is

$$O(V) = \{\alpha \in \text{End}(V) : \alpha \text{ is orthogonal}\}.$$

**Definition** (Unitary map). Let  $V$  be a finite-dimensional complex vector space. Then  $\alpha \in \text{End}(V)$  is *unitary* if

$$(\alpha(\mathbf{v}), \alpha(\mathbf{w})) = (\mathbf{v}, \mathbf{w})$$

for all  $\mathbf{v}, \mathbf{w} \in V$ .

**Definition** (Unitary group). Let  $V$  be a finite-dimensional complex inner product space. Then the *unitary group* of  $V$  is

$$U(V) = \{\alpha \in \text{End}(V) : \alpha \text{ is unitary}\}.$$

## 8.4 Spectral theory