Part IB — Linear Algebra
Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Definition of a vector space (over $\mathbb{R}$ or $\mathbb{C}$), subspaces, the space spanned by a subset. Linear independence, bases, dimension. Direct sums and complementary subspaces. [3]

Linear maps, isomorphisms. Relation between rank and nullity. The space of linear maps from U to V, representation by matrices. Change of basis. Row rank and column rank. [4]

Determinant and trace of a square matrix. Determinant of a product of two matrices and of the inverse matrix. Determinant of an endomorphism. The adjugate matrix. [3]

Eigenvalues and eigenvectors. Diagonal and triangular forms. Characteristic and minimal polynomials. Cayley-Hamilton Theorem over $\mathbb{C}$. Algebraic and geometric multiplicity of eigenvalues. Statement and illustration of Jordan normal form. [4]

Dual of a finite-dimensional vector space, dual bases and maps. Matrix representation, rank and determinant of dual map. [2]

Bilinear forms. Matrix representation, change of basis. Symmetric forms and their link with quadratic forms. Diagonalisation of quadratic forms. Law of inertia, classification by rank and signature. Complex Hermitian forms. [4]

Inner product spaces, orthonormal sets, orthogonal projection, $V = W \oplus W^{\perp}$. Gram-Schmidt orthogonalisation. Adjoints. Diagonalisation of Hermitian matrices. Orthogonality of eigenvectors and properties of eigenvalues. [4]

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1.1 Definitions and examples

Notation. We will use $\mathbb{F}$ to denote an arbitrary field, usually $\mathbb{R}$ or $\mathbb{C}$.

Definition (Vector space)**.** An $\mathbb{F}$-*vector space* is an (additive) abelian group V together with a function $\mathbb{F} \times V \to V$, written $(\lambda, \mathbf{v}) \mapsto \lambda\mathbf{v}$, such that

(i) $\lambda(\mu\mathbf{v}) = \lambda\mu\mathbf{v}$ for all $\lambda, \mu \in \mathbb{F}$, $\mathbf{v} \in V$ (associativity)

(ii) $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$ for all $\lambda \in \mathbb{F}$, $\mathbf{u}, \mathbf{v} \in V$ (distributivity in V)

(iii) $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$ for all $\lambda, \mu \in \mathbb{F}$, $\mathbf{v} \in V$ (distributivity in $\mathbb{F}$)

(iv) $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$ (identity)

We always write $\mathbf{0}$ for the additive identity in V, and call this the identity. By abuse of notation, we also write 0 for the trivial vector space $\{\mathbf{0}\}$.

Definition (Subspace)**.** If V is an $\mathbb{F}$-vector space, then $U \subseteq V$ is an ($\mathbb{F}$-linear) *subspace* if

(i) $\mathbf{u}, \mathbf{v} \in U$ implies $\mathbf{u} + \mathbf{v} \in U$.

(ii) $\mathbf{u} \in U, \lambda \in \mathbb{F}$ implies $\lambda\mathbf{u} \in U$.

(iii) $\mathbf{0} \in U$.

These conditions can be expressed more concisely as "U is non-empty and if $\lambda, \mu \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in U$, then $\lambda\mathbf{u} + \mu\mathbf{v} \in U$".

Alternatively, U is a subspace of V if it is itself a vector space, inheriting the operations from V.

We sometimes write $U \leq V$ if U is a subspace of V.

Definition (Sum of subspaces)**.** Suppose U, W are subspaces of an $\mathbb{F}$ vector space V. The *sum* of U and V is

$$U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}.$$

Definition (Quotient spaces)**.** Let V be a vector space, and $U \subseteq V$ a subspace. Then the quotient group V/U can be made into a vector space called the *quotient space*, where scalar multiplication is given by $(\lambda, \mathbf{v} + U) = (\lambda\mathbf{v}) + U$.

This is well defined since if $\mathbf{v} + U = \mathbf{w} + U \in V/U$, then $\mathbf{v} - \mathbf{w} \in U$. Hence for $\lambda \in \mathbb{F}$, we have $\lambda\mathbf{v} - \lambda\mathbf{w} \in U$. So $\lambda\mathbf{v} + U = \lambda\mathbf{w} + U$.

1.2 Linear independence, bases and the Steinitz exchange lemma

Definition (Span)**.** Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$. The *span* of S is defined as

$$\langle S \rangle = \left\{\sum_{i=1}^n \lambda_i \mathbf{s}_i : \lambda_i \in \mathbb{F}, \mathbf{s}_i \in S, n \geq 0\right\}$$

This is the smallest subspace of V containing S.

Note that the sums must be finite. We will not play with infinite sums, since the notion of convergence is not even well defined in a general vector space.

Definition (Spanning set)**.** Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$. S *spans* V if $\langle S \rangle = V$.

Definition (Linear independence)**.** Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$. Then S is *linearly independent* if whenever

$$\sum_{i=1}^{n} \lambda_i \mathbf{s}_i = \mathbf{0} \text{ with } \lambda_i \in \mathbb{F}, \mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_n \in S \text{ distinct},$$

we must have $\lambda_i = 0$ for all i.

If S is not linearly independent, we say it is *linearly dependent*.

Definition (Basis)**.** Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$. Then S is a *basis* for V if S is linearly independent and spans V.

Definition (Finite dimensional)**.** A vector space is *finite dimensional* if there is a finite basis.

Definition (Dimension)**.** If V is a vector space over $\mathbb{F}$ with finite basis S, then the *dimension* of V, written

$$\dim V = \dim_{\mathbb{F}} V = |S|.$$

1.3 Direct sums

Definition ((Internal) direct sum)**.** Suppose V is a vector space over $\mathbb{F}$ and $U, W \subseteq V$ are subspaces. We say that V is the *(internal) direct sum* of U and W if

(i) $U + W = V$

(ii) $U \cap W = 0$.

We write $V = U \oplus W$.

Equivalently, this requires that every $\mathbf{v} \in V$ can be written uniquely as $\mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U, \mathbf{w} \in W$. We say that U and W are *complementary subspaces* of V.

Definition ((External) direct sum)**.** If U, W are vector spaces over $\mathbb{F}$, the *(external) direct sum* is

$$U \oplus W = \{(\mathbf{u}, \mathbf{w}) : \mathbf{u} \in U, \mathbf{w} \in W\},$$

with addition and scalar multiplication componentwise:

$$(\mathbf{u}_1, \mathbf{w}_1) + (\mathbf{u}_2, \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{w}_1 + \mathbf{w}_2), \quad \lambda(\mathbf{u}, \mathbf{w}) = (\lambda \mathbf{u}, \lambda \mathbf{w}).$$

Definition ((Multiple) (internal) direct sum)**.** If $U_1, \cdots, U_n \subseteq V$ are subspaces of V, then V is the *(internal) direct sum*

$$V = U_1 \oplus \cdots \oplus U_n = \bigoplus_{i=1}^{n} U_i$$

if every $\mathbf{v} \in V$ can be written uniquely as $\mathbf{v} = \sum \mathbf{u}_i$ with $\mathbf{u}_i \in U_i$.

This can be extended to an infinite sum with the same definition, just noting that the sum $\mathbf{v} = \sum \mathbf{u}_i$ has to be finite.

Definition ((Multiple) (external) direct sum)**.** If $U_1, \cdots, U_n$ are vector spaces over $\mathbb{F}$, the external direct sum is

$$U_1 \oplus \cdots \oplus U_n = \bigoplus_{i=1}^n U_i = \{(\mathbf{u}_1, \cdots, \mathbf{u}_n) : \mathbf{u}_i \in U_i\},$$

with pointwise operations.

This can be made into an infinite sum if we require that all but finitely many of the $\mathbf{u}_i$ have to be zero.

2 Linear maps

2.1 Definitions and examples

Definition (Linear map)**.** Let U, V be vector spaces over $\mathbb{F}$. Then $\alpha : U \to V$ is a *linear map* if

(i) $\alpha(\mathbf{u}_1 + \mathbf{u}_2) = \alpha(\mathbf{u}_1) + \alpha(\mathbf{u}_2)$ for all $\mathbf{u}_i \in U$.

(ii) $\alpha(\lambda \mathbf{u}) = \lambda\alpha(\mathbf{u})$ for all $\lambda \in \mathbb{F}, \mathbf{u} \in U$.

We write $\mathcal{L}(U, V)$ for the set of linear maps $U \to V$.

Definition (Isomorphism)**.** We say a linear map $\alpha : U \to V$ is an *isomorphism* if there is some $\beta : V \to U$ (also linear) such that $\alpha \circ \beta = \mathrm{id}_V$ and $\beta \circ \alpha = \mathrm{id}_U$.

If there exists an isomorphism $U \to V$, we say U and V are *isomorphic*, and write $U \cong V$.

Definition (Image and kernel)**.** Let $\alpha : U \to V$ be a linear map. Then the *image* of α is

$$\operatorname{im} \alpha = \{\alpha(\mathbf{u}) : \mathbf{u} \in U\}.$$

The *kernel* of α is

$$\ker \alpha = \{\mathbf{u} : \alpha(\mathbf{u}) = \mathbf{0}\}.$$

2.2 Linear maps and matrices

Definition (Matrix representation)**.** We call the matrix corresponding to a linear map $\alpha \in \mathcal{L}(U, V)$ under the corollary the *matrix representing* α with respect to the bases $(\mathbf{e}_1, \cdots, \mathbf{e}_m)$ and $(\mathbf{f}_1, \cdots, \mathbf{f}_n)$.

2.3 The first isomorphism theorem and the rank-nullity theorem

Definition (Rank and nullity)**.** If $\alpha : U \to V$ is a linear map between finite-dimensional vector spaces over $\mathbb{F}$ (in fact we just need U to be finite-dimensional), the *rank* of α is the number $r(\alpha) = \dim \operatorname{im} \alpha$. The *nullity* of α is the number $n(\alpha) = \dim \ker \alpha$.

2.4 Change of basis

Definition (Equivalent matrices)**.** We say $A, B \in \mathrm{Mat}_{n,m}(\mathbb{F})$ are *equivalent* if there are invertible matrices $P \in \mathrm{Mat}_m(\mathbb{F})$, $Q \in \mathrm{Mat}_n(\mathbb{F})$ such that $B = Q^{-1}AP$.

Definition (Column and row rank)**.** If $A \in \mathrm{Mat}_{n,m}(\mathbb{F})$, then

- The *column rank* of A, written $r(A)$, is the dimension of the subspace of $\mathbb{F}^n$ spanned by the columns of A.
- The *row rank* of A, written $r(A)$, is the dimension of the subspace of $\mathbb{F}^m$ spanned by the rows of A. Alternatively, it is the column rank of A^T.

2.5 Elementary matrix operations

Definition (Elementary matrices)**.** We call the following matrices of $\mathrm{GL}_n(\mathbb{F})$ *elementary matrices*:

$$S_{ij}^n = \begin{pmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 0 & & & & 1 & & \\ & & & & 1 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 1 & & & \\ & & & 1 & & & & 0 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix}$$

This is called a reflection, where the rows we changed are the ith and jth row.

$$E_{ij}^n(\lambda) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \lambda & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

This is called a shear, where λ appears at the i, jth entry.

$$T_i^n(\lambda) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \lambda & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

This is called a shear, where $\lambda \neq 0$ appears at the ith column.

3 Duality

3.1 Dual space

Definition (Dual space)**.** Let V be a vector space over $\mathbb{F}$. The *dual* of V is defined as
$$V^* = \mathcal{L}(V, \mathbb{F}) = \{\theta : V \to \mathbb{F} : \theta \text{ linear}\}.$$
Elements of V^* are called *linear functionals* or *linear forms*.

Definition (Annihilator)**.** Let $U \subseteq V$. Then the *annihilator* of U is
$$U^0 = \{\theta \in V^* : \theta(\mathbf{u}) = 0, \forall \mathbf{u} \in U\}.$$
If $W \subseteq V^*$, then the *annihilator* of W is
$$W^0 = \{\mathbf{v} \in V : \theta(\mathbf{v}) = 0, \forall \theta \in W\}.$$

3.2 Dual maps

Definition (Dual map)**.** Let V, W be vector spaces over $\mathbb{F}$ and $\alpha : V \to W \in \mathcal{L}(V, W)$. The *dual map* to α, written $\alpha^* : W^* \to V^*$ is given by $\theta \mapsto \theta \circ \alpha$. Since the composite of linear maps is linear, $\alpha^*(\theta) \in V^*$. So this is a genuine map.

4 Bilinear forms I

Definition (Bilinear form)**.** Let V, W be vector spaces over $\mathbb{F}$. Then a function $\phi : V \times W \to \mathbb{F}$ is a *bilinear form* if it is linear in each variable, i.e. for each $\mathbf{v} \in V$, $\phi(\mathbf{v}, \cdot\,) : W \to \mathbb{F}$ is linear; for each $\mathbf{w} \in W$, $\phi(\,\cdot\,, \mathbf{w}) : V \to \mathbb{F}$ is linear.

Definition (Matrix representing bilinear form)**.** Let $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ be a basis for V and $(\mathbf{f}_1, \cdots, \mathbf{f}_m)$ be a basis for W, and $\psi : V \times W \to \mathbb{F}$. Then the *matrix A representing ψ* with respect to the basis is defined to be

$$A_{ij} = \psi(\mathbf{e}_i, \mathbf{f}_j).$$

Definition (Left and right kernel)**.** The kernel of ψ_L is *left kernel* of ψ, while the kernel of ψ_R is the *right kernel* of ψ.

Definition (Non-degenerate bilinear form)**.** ψ is *non-degenerate* if the left and right kernels are both trivial. We say ψ is *degenerate* otherwise.

Definition (Rank of bilinear form)**.** If $\psi : V \to W$ is a bilinear form $\mathbb{F}$ on a finite-dimensional vector space V, then the *rank* of V is the rank of any matrix representing ϕ. This is well-defined since $r(P^T AQ) = r(A)$ if P and Q are invertible.

Alternatively, it is the rank of ψ_L (or ψ_R).

5 Determinants of matrices

Definition (Determinant)**.** Let $A \in \mathrm{Mat}_{n,n}(\mathbb{F})$. Its *determinant* is

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^n A_{i\sigma(i)}.$$

Definition (Volume form)**.** A *volume form* on $\mathbb{F}^n$ is a function $d: \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ that is

(i) Multilinear, i.e. for all i and all $\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_n \in \mathbb{F}^n$, we have

$$\mathrm{d}(\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}, \cdot, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_n) \in (\mathbb{F}^n)^*.$$

(ii) Alternating, i.e. if $\mathbf{v}_i = \mathbf{v}_j$ for some $i \neq j$, then

$$d(\mathbf{v}_1, \cdots, \mathbf{v}_n) = 0.$$

Definition (Singular matrices)**.** A matrix A is *singular* if $\det A = 0$. Otherwise, it is *non-singular*.

Notation. Write $\hat{A}_{ij}$ for the matrix obtained from A by deleting the ith row and jth column.

Definition (Adjugate matrix)**.** Let $A \in \mathrm{Mat}_n(\mathbb{F})$. The *adjugate matrix* of A, written $\operatorname{adj} A$, is the $n \times n$ matrix such that $(\operatorname{adj} A)_{ij} = (-1)^{i+j} \det \hat{A}_{ji}$.

6 Endomorphisms

6.1 Invariants

Definition. If V is a (finite-dimensional) vector space over $\mathbb{F}$. An *endomorphism* of V is a linear map $\alpha : V \to V$. We write $\mathrm{End}(V)$ for the $\mathbb{F}$-vector space of all such linear maps, and I for the identity map $V \to V$.

Definition (Similar matrices)**.** We say matrices A and B are *similar* or *conjugate* if there is some P invertible such that $B = P^{-1}AP$.

Definition (Trace)**.** The *trace* of a matrix of $A \in \mathrm{Mat}_n(\mathbb{F})$ is defined by

$$\operatorname{tr} A = \sum_{i=1}^{n} A_{ii}.$$

Definition (Trace and determinant of endomorphism)**.** Let $\alpha \in \mathrm{End}(V)$, and A be a matrix representing α under any basis. Then the *trace* of α is $\operatorname{tr}\alpha = \operatorname{tr} A$, and the *determinant* is $\det\alpha = \det A$.

Definition (Eigenvalue and eigenvector)**.** Let $\alpha \in \mathrm{End}(V)$. Then $\lambda \in \mathbb{F}$ is an *eigenvalue* (or E-value) if there is some $\mathbf{v} \in V \setminus \{0\}$ such that $\alpha\mathbf{v} = \lambda\mathbf{v}$.

$\mathbf{v}$ is an *eigenvector* if $\alpha(\mathbf{v}) = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{F}$.

When $\lambda \in \mathbb{F}$, the λ-*eigenspace*, written $E_\alpha(\lambda)$ or $E(\lambda)$ is the subspace of V containing all the λ-eigenvectors, i.e.

$$E_\alpha(\lambda) = \ker(\lambda\iota - \alpha).$$

where ι is the identity function.

Definition (Characteristic polynomial)**.** The *characteristic polynomial* of α is defined by

$$\chi_\alpha(t) = \det(t\iota - \alpha).$$

Definition (Diagonalizable)**.** We say $\alpha \in \mathrm{End}(V)$ is diagonalizable if there is some basis for V such that α is represented by a diagonal matrix, i.e. all terms not on the diagonal are zero.

6.2 The minimal polynomial

6.2.1 Aside on polynomials

Definition (Polynomial)**.** A *polynomial* over $\mathbb{F}$ is an object of the form

$$f(t) = a_m t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0,$$

with $m \geq 0, a_0, \cdots, a_m \in \mathbb{F}$.

We write $\mathbb{F}[t]$ for the set of polynomials over $\mathbb{F}$.

Definition (Degree)**.** Let $f \in \mathbb{F}[t]$. Then the *degree* of f, written $\deg f$ is the largest n such that $a_n \neq 0$. In particular, $\deg 0 = -\infty$.

Definition (Multiplicity of a root)**.** Let $f \in \mathbb{F}[t]$ and λ a root of f. We say λ has *multiplicity* k if $(t-\lambda)^k$ is a factor of f but $(t-\lambda)^{k+1}$ is not, i.e.

$$f(t) = (t-\lambda)^k g(t)$$

for some $g(t) \in \mathbb{F}[t]$ with $g(\lambda) \neq 0$.

6.2.2 Minimal polynomial

Notation. Given $f(t) = \sum_{i=0}^m a_i t^i \in \mathbb{F}[t]$, $A \in \mathrm{Mat}_n(\mathbb{F})$ and $\alpha \in \mathrm{End}(V)$, we can write
$$f(A) = \sum_{i=0}^m a_i A^i, \quad f(\alpha) = \sum_{i=0}^m a_i \alpha^i$$
where $A^0 = I$ and $\alpha^0 = \iota$.

Definition (Minimal polynomial)**.** The *minimal polynomial* of $\alpha \in \mathrm{End}(V)$ is the non-zero monic polynomial $M_\alpha(t)$ of least degree such that $M_\alpha(\alpha) = 0$.

6.3 The Cayley-Hamilton theorem

Definition (Triangulable)**.** An endomorphism $\alpha \in \mathrm{End}(V)$ is *triangulable* if there is a basis for V such that α is represented by an upper triangular matrix
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

6.4 Multiplicities of eigenvalues and Jordan normal form

Definition (Algebraic and geometry multiplicity)**.** Let $\alpha \in \mathrm{End}(V)$ and λ an eigenvalue of α. Then

(i) The *algebraic multiplicity* of λ, written a_λ, is the multiplicity of λ as a root of $\chi_\alpha(t)$.

(ii) The *geometric multiplicity* of λ, written g_λ, is the dimension of the corresponding eigenspace, $\dim E_\alpha(\lambda)$.

(iii) c_λ is the multiplicity of λ as a root of the minimal polynomial $m_\alpha(t)$.

Definition (Jordan normal form)**.** We say $A \in \mathrm{Mat}_N(\mathbb{C})$ is in *Jordan normal form* if it is a block diagonal of the form
$$\begin{pmatrix} J_{n_1}(\lambda_1) & & & 0 \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{n_k}(\lambda_k) \end{pmatrix}$$
where $k \geq 1$, $n_1, \cdots, n_k \in \mathbb{N}$ such that $n = \sum n_i$, $\lambda_1, \cdots, \lambda_k$ not necessarily distinct, and
$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$$
is an $m \times m$ matrix. Note that $J_m(\lambda) = \lambda I_m + J_m(0)$.

Definition (Nilpotent)**.** We say $\alpha \in \mathrm{End}(V)$ is nilpotent if there is some r such that $\alpha^r = 0$.

7 Bilinear forms II

7.1 Symmetric bilinear forms and quadratic forms

Definition (Symmetric bilinear form)**.** Let V is a vector space over $\mathbb{F}$. A bilinear form $\phi : V \times V \to \mathbb{F}$ is *symmetric* if

$$\phi(\mathbf{v}, \mathbf{w}) = \phi(\mathbf{w}, \mathbf{v})$$

for all $\mathbf{v}, \mathbf{w} \in V$.

Definition (Congruent matrices)**.** Two square matrices A, B are *congruent* if there exists some invertible P such that

$$B = P^T AP.$$

Definition (Quadratic form)**.** A *function* $q : V \to \mathbb{F}$ is a *quadratic form* if there exists some bilinear form ϕ such that

$$q(\mathbf{v}) = \phi(\mathbf{v}, \mathbf{v})$$

for all $\mathbf{v} \in V$.

Definition (Positive/negative (semi-)definite)**.** Let ϕ be a symmetric bilinear form on a finite-dimensional real vector space V. We say

(i) ϕ is *positive definite* if $\phi(\mathbf{v}, \mathbf{v}) > 0$ for all $\mathbf{v} \in V \setminus \{0\}$.

(ii) ϕ is *positive semi-definite* if $\phi(\mathbf{v}, \mathbf{v}) \geq 0$ for all $\mathbf{v} \in V$.

(iii) ϕ is *negative definite* if $\phi(\mathbf{v}, \mathbf{v}) < 0$ for all $\mathbf{v} \in V \setminus \{0\}$.

(iv) ϕ is *negative semi-definite* if $\phi(\mathbf{v}, \mathbf{v}) \leq 0$ for all $\mathbf{v} \in V$.

Definition (Signature)**.** The *signature* of a bilinear form ϕ is the number $p - q$, where p and q are as above.

7.2 Hermitian form

Definition (Sesquilinear form)**.** Let V, W be complex vector spaces. Then a *sesquilinear form* is a function $\phi : V \times W \to \mathbb{C}$ such that

(i) $\phi(\lambda \mathbf{v}_1 + \mu \mathbf{v}_2, \mathbf{w}) = \bar{\lambda}\phi(\mathbf{v}_1, \mathbf{w}) + \bar{\mu}\phi(\mathbf{v}_2, \mathbf{w})$.

(ii) $\phi(\mathbf{v}, \lambda \mathbf{w}_1 + \mu \mathbf{w}_2) = \lambda\phi(\mathbf{v}, \mathbf{w}_1) + \mu\phi(\mathbf{v}\mathbf{w}_2)$.

for all $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V$, $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in W$ and $\lambda, \mu \in \mathbb{C}$.

Definition (Representation of sesquilinear form)**.** Let V, W be finite-dimensional complex vector spaces with basis $(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ and $(\mathbf{w}_1, \cdots, \mathbf{w}_m)$ respectively, and $\phi : V \times W \to \mathbb{C}$ be a sesquilinear form. Then the matrix representing ϕ with respect to these bases is

$$A_{ij} = \phi(\mathbf{v}_i, \mathbf{w}_j).$$

for $1 \leq i \leq n, 1 \leq j \leq m$.

Definition (Hermitian sesquilinear form)**.** A sesquilinear form on $V \times V$ is *Hermitian* if

$$\phi(\mathbf{v}, \mathbf{w}) = \overline{\phi(\mathbf{w}, \mathbf{v})}.$$

8 Inner product spaces

8.1 Definitions and basic properties

Definition (Inner product space)**.** Let V be a vector space. An *inner product* on V is a positive-definite symmetric bilinear/hermitian form. We usually write (x, y) instead of $\phi(x, y)$.

A vector space equipped with an inner product is an *inner product space.*

Definition (Orthogonal vectors)**.** Let V be an inner product space. Then $\mathbf{v}, \mathbf{w} \in V$ are *orthogonal* if $(\mathbf{v}, \mathbf{w}) = 0$.

Definition (Orthonormal set)**.** Let V be an inner product space. A set $\{\mathbf{v}_i : i \in I\}$ is an *orthonormal set* if for any $i, j \in I$, we have

$$(\mathbf{v}_i, \mathbf{v}_j) = \delta_{ij}$$

Definition (Orthonormal basis)**.** Let V be an inner product space. A subset of V is an *orthonormal basis* if it is an orthonormal set and is a basis.

8.2 Gram-Schmidt orthogonalization

Definition (Orthogonal internal direct sum)**.** Let V be an inner product space and $V_1, V_2 \leq V$. Then V is the *orthogonal internal direct sum* of V_1 and V_2 if it is a direct sum and V_1 and V_2 are orthogonal. More precisely, we require

(i) $V = V_1 + V_2$

(ii) $V_1 \cap V_2 = 0$

(iii) $(\mathbf{v}_1, \mathbf{v}_2) = 0$ for all $\mathbf{v}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$.

Note that condition (iii) implies (ii), but we write it for the sake of explicitness.

We write $V = V_1 \perp V_2$.

Definition (Orthogonal complement)**.** If $W \leq V$ is a subspace of an inner product space V, then the *orthogonal complement* of W in V is the subspace

$$W^\perp = \{\mathbf{v} \in V : (\mathbf{v}, \mathbf{w}) = 0, \forall \mathbf{w} \in W\}.$$

Definition (Orthogonal external direct sum)**.** Let V_1, V_2 be inner product spaces. The *orthogonal external direct sum* of V_1 and V_2 is the vector space $V_1 \oplus V_2$ with the inner product defined by

$$(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2) = (\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2),$$

with $\mathbf{v}_1, \mathbf{w}_1 \in V_1$, $\mathbf{v}_2, \mathbf{w}_2 \in V_2$.

Here we write $\mathbf{v}_1 + \mathbf{v}_2 \in V_1 \oplus V_2$ instead of $(\mathbf{v}_1, \mathbf{v}_2)$ to avoid confusion.

8.3 Adjoints, orthogonal and unitary maps

Definition (Adjoint)**.** We call the map α^* the *adjoint* of α.

Definition (Self-adjoint)**.** Let V be an inner product space, and $\alpha \in \mathrm{End}(V)$. Then α is *self-adjoint* if $\alpha = \alpha^*$, i.e.

$$(\alpha(\mathbf{v}), \mathbf{w}) = (\mathbf{v}, \alpha(\mathbf{w}))$$

for all $\mathbf{v}, \mathbf{w}$.

Definition (Orthogonal endomorphism)**.** Let V be a real inner product space. Then $\alpha \in \mathrm{End}(V)$ is *orthogonal* if

$$(\alpha(\mathbf{v}), \alpha(\mathbf{w})) = (\mathbf{v}, \mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in V$.

Definition (Orthogonal group)**.** Let V be a real inner product space. Then the *orthogonal group* of V is

$$\mathrm{O}(V) = \{\alpha \in \mathrm{End}(V) : \alpha \text{ is orthogonal}\}.$$

Definition (Unitary map)**.** Let V be a finite-dimensional complex vector space. Then $\alpha \in \mathrm{End}(V)$ is *unitary* if

$$(\alpha(\mathbf{v}), \alpha(\mathbf{w})) = (\mathbf{v}, \mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in V$.

Definition (Unitary group)**.** Let V be a finite-dimensional complex inner product space. Then the *unitary group* of V is

$$U(V) = \{\alpha \in \mathrm{End}(V) : \alpha \text{ is unitary}\}.$$

8.4 Spectral theory