Part IB — Analysis II

Theorems

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Uniform convergence
The general principle of uniform convergence. A uniform limit of continuous functions is continuous. Uniform convergence and termwise integration and differentiation of series of real-valued functions. Local uniform convergence of power series. [3]

Uniform continuity and integration
Continuous functions on closed bounded intervals are uniformly continuous. Review of basic facts on Riemann integration (from Analysis I). Informal discussion of integration of complex-valued and $\mathbb{R}^n$-valued functions of one variable; proof that $\| \int_a^b f(x)\,\mathrm{d}x\| \leq \int_a^b \|f(x)\|\,\mathrm{d}x$. [2]

$\mathbb{R}^n$ as a normed space
Definition of a normed space. Examples, including the Euclidean norm on $\mathbb{R}^n$ and the uniform norm on $C[a, b]$. Lipschitz mappings and Lipschitz equivalence of norms. The Bolzano-Weierstrass theorem in $\mathbb{R}^n$. Completeness. Open and closed sets. Continuity for functions between normed spaces. A continuous function on a closed bounded set in $\mathbb{R}^n$ is uniformly continuous and has closed bounded image. All norms on a finite-dimensional space are Lipschitz equivalent. [5]

Differentiation from $\mathbb{R}^m$ to $\mathbb{R}^n$
Definition of derivative as a linear map; elementary properties, the chain rule. Partial derivatives; continuous partial derivatives imply differentiability. Higher-order derivatives; symmetry of mixed partial derivatives (assumed continuous). Taylor's theorem. The mean value inequality. Path-connectedness for subsets of $\mathbb{R}^n$; a function having zero derivative on a path-connected open subset is constant. [6]

Metric spaces
Definition and examples. *Metrics used in Geometry*. Limits, continuity, balls, neighbourhoods, open and closed sets. [4]

The Contraction Mapping Theorem
The contraction mapping theorem. Applications including the inverse function theorem (proof of continuity of inverse function, statement of differentiability). Picard's solution of differential equations. [4]

Contents

0 Introduction

1 Uniform convergence

Theorem. Let $f_n : E \to \mathbb{R}$ be a sequence of functions. Then (f_n) converges uniformly if and only if (f_n) is uniformly Cauchy.

Theorem (Uniform convergence and continuity)**.** Let $E \subseteq \mathbb{R}$, $x \in E$ and $f_n, f : E \to \mathbb{R}$. Suppose $f_n \to f$ uniformly, and f_n are continuous at x for all n. Then f is also continuous at x.

In particular, if f_n are continuous everywhere, then f is continuous everywhere.

Theorem (Uniform convergence and integrals)**.** Let $f_n, f : [a, b] \to \mathbb{R}$ be Riemann integrable, with $f_n \to f$ uniformly. Then

$$\int_a^b f_n(t)\;\mathrm{d}t \to \int_a^b f(t)\;\mathrm{d}t.$$

Theorem. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of functions differentiable on $[a, b]$ (at the end points a, b, this means that the one-sided derivatives exist). Suppose the following holds:

(i) For some $c \in [a, b]$, $f_n(c)$ converges.

(ii) The sequence of derivatives (f_n') converges uniformly on $[a, b]$.

Then (f_n) converges uniformly on $[a, b]$, and if $f = \lim f_n$, then f is differentiable with derivative $f'(x) = \lim f_n'(x)$.

Proposition.

(i) Let $f_n, g_n : E \to \mathbb{R}$, be sequences, and $f_n \to f$, $g_n \to g$ uniformly on E. Then for any $a, b \in \mathbb{R}$, $af_n + bg_n \to af + bg$ uniformly.

(ii) Let $f_n \to f$ uniformly, and let $g : E \to \mathbb{R}$ is bounded. Then $gf_n : E \to \mathbb{R}$ converges uniformly to gf.

2 Series of functions

2.1 Convergence of series

Proposition. Let $g_n : E \to \mathbb{R}$. If $\sum g_n$ converges absolutely uniformly, then $\sum g_n$ converges uniformly.

Theorem (Weierstrass M-test)**.** Let $g_n : E \to \mathbb{R}$ be a sequence of functions. Suppose there is some sequence M_n such that for all n, we have

$$\sup_{x \in E} |g_n(x)| \leq M_n.$$

If $\sum M_n$ converges, then $\sum g_n$ converges absolutely uniformly.

2.2 Power series

Theorem. Let $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ be a real power series. Then there exists a unique number $R \in [0, +\infty]$ (called the radius of convergence) such that

(i) If $|x - a| < R$, then $\sum c_n(x-a)^n$ converges absolutely.

(ii) If $|x - a| > R$, then $\sum c_n(x-a)^n$ diverges.

(iii) If $R > 0$ and $0 < r < R$, then $\sum c_n(x-a)^n$ converges absolutely uniformly on $[a - r, a + r]$.

We say that the sum converges locally absolutely uniformly inside the circle of convergence, i.e. for every point $y \in (a - R, a + R)$, there is some open interval around y on which the sum converges absolutely uniformly.

These results hold for complex power series as well, but for concreteness we will just do it for real series.

Theorem (Termwise differentiation of power series)**.** Suppose $\sum c_n(x-n)^n$ is a real power series with radius of convergence $R > 0$. Then

(i) The "derived series"

$$\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

has radius of convergence R.

(ii) The function defined by $f(x) = \sum c_n(x-a)^n$, $x \in (a - R, a + R)$ is differentiable with derivative $f'(x) = \sum nc_n(x-a)^{n-1}$ within the (open) circle of convergence.

3 Uniform continuity and integration

3.1 Uniform continuity

Theorem. Any continuous function on a closed, bounded interval is uniformly continuous.

3.2 Applications to Riemann integrability

Theorem (Riemann criterion for integrability)**.** A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every ε, there is a partition P such that

$$U(P, f) - L(P, f) < \varepsilon.$$

Theorem. If $f : [a, b] \to [A, B]$ is integrable and $g : [A, B] \to \mathbb{R}$ is continuous, then $g \circ f : [a, b] \to \mathbb{R}$ is integrable.

Corollary. A continuous function $g : [a, b] \to \mathbb{R}$ is integrable.

Theorem. Let $f_n : [a, b] \to \mathbb{R}$ be bounded and integrable for all n. Then if (f_n) converges uniformly to a function $f : [a, b] \to \mathbb{R}$, then f is bounded and integrable.

Proposition. If $\mathbf{f} : [a, b] \to \mathbb{R}^n$ is integrable, then the function $\|\mathbf{f}\| : [a, b] \to \mathbb{R}$ defined by

$$\|\mathbf{f}\|(x) = \|\mathbf{f}(x)\| = \sqrt{\sum_{j=1}^{n} f_j^2(x)}.$$

is integrable, and

$$\left\| \int_a^b \mathbf{f}(x)\, \mathrm{d}x \right\| \leq \int_a^b \|\mathbf{f}\|(x)\, \mathrm{d}x.$$

3.3 Non-examinable fun*

Theorem (Weierstrass Approximation Theorem*)**.** If $f : [0, 1] \to \mathbb{R}$ is continuous, then there exists a sequence of polynomials (p_n) such that $p_n \to f$ uniformly. In fact, the sequence can be given by

$$p_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}.$$

These are known as *Bernstein polynomials*.

Theorem (Lebesgue's theorem on the Riemann integral*)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $\mathcal{D}_f$ be the set of points of discontinuities of f. Then f is Riemann integrable *if and only if* $\mathcal{D}_f$ has measure zero.

4 $\mathbb{R}^n$ as a normed space

4.1 Normed spaces

Lemma (Cauchy-Schwarz inequality (for integrals))**.** If $f, g \in C([a, b])$, $f, g \geq 0$, then
$$\int_a^b fg \, \mathrm{d}x \leq \left(\int_a^b f^2 \, \mathrm{d}x)\right)^{1/2} \left(\int_a^b g^2 \, \mathrm{d}x\right)^{1/2}.$$

Proposition. If $\|\cdot\|$ and $\|\cdot\|'$ are Lipschitz equivalent norms on a vector space V, then

(i) A subset $E \subseteq V$ is bounded with respect to $\|\cdot\|$ if and only if it is bounded with respect to $\|\cdot\|'$.

(ii) A sequence x_k converges to x with respect to $\|\cdot\|$ if and only if it converges to x with respect to $\|\cdot\|'$.

Proposition. Let $(V, \|\cdot\|)$ be a normed space. Then

(i) If $\mathbf{x}_k \to \mathbf{x}$ and $\mathbf{x}_k \to \mathbf{y}$, then $\mathbf{x} = \mathbf{y}$.

(ii) If $\mathbf{x}_k \to \mathbf{x}$, then $a\mathbf{x}_k \to a\mathbf{x}$.

(iii) If $\mathbf{x}_k \to \mathbf{x}$, $\mathbf{y}_k \to \mathbf{y}$, then $\mathbf{x}_k + \mathbf{y}_k \to \mathbf{x} + \mathbf{y}$.

Proposition. Convergence in $\mathbb{R}^n$ (with respect to, say, the Euclidean norm) is equivalent to coordinate-wise convergence, i.e. $\mathbf{x}^{(k)} \to \mathbf{x}$ if and only if $x_j^{(k)} \to x_j$ for all j.

Theorem (Bolzano-Weierstrass theorem in $\mathbb{R}^n$)**.** Any bounded sequence in $\mathbb{R}^n$ (with, say, the Euclidean norm) has a convergent subsequence.

4.2 Cauchy sequences and completeness

Proposition. Any convergent sequence is Cauchy.

Proposition. A Cauchy sequence is bounded.

Proposition. If a Cauchy sequence has a subsequence converging to an element $\mathbf{x}$, then the whole sequence converges to $\mathbf{x}$.

Proposition. If $\|\cdot\|'$ is Lipschitz equivalent to $\|\cdot\|$ on V, then $(\mathbf{x}_k)$ is Cauchy with respect to $\|\cdot\|$ if and only if $(\mathbf{x}_k)$ is Cauchy with respect to $\|\cdot\|'$. Also, $(V, \|\cdot\|)$ is complete if and only if $(V, \|\cdot\|')$ is complete.

Theorem. $\mathbb{R}^n$ (with the Euclidean norm, say) is complete.

Proposition. $B_r(\mathbf{y}) \subseteq V$ is an open subset for all $r > 0$, $\mathbf{y} \in V$.

Proposition. Let $E \subseteq V$. Then E contains all of its limit points if and only if $V \setminus E$ is open in V.

Lemma. Let $(V, \|\cdot\|)$ be a normed space, E any subset of V. Then a point $\mathbf{y} \in V$ is a limit point of E if and only if

$$(B_r(\mathbf{y}) \setminus \{\mathbf{y}\}) \cap E \neq \emptyset$$

for every r.

Proposition. Let $E \subseteq V$. Then E contains all of its limit points if and only if $V \setminus E$ is open in V.

4.3 Sequential compactness

Theorem. Let $(V, \|\cdot\|)$ be a normed vector space, $K \subseteq V$ a subset. Then

(i) If K is compact, then K is closed and bounded.

(ii) If V is $\mathbb{R}^n$ (with, say, the Euclidean norm), then if K is closed and bounded, then K is compact.

4.4 Mappings between normed spaces

Theorem. Let $(V, \|\cdot\|)$, $(V', \|\cdot\|')$ be normed spaces, $E \subseteq V$, $f : E \to \mathbf{V}'$. Then f is continuous at $\mathbf{y} \in E$ if and only if for any sequence $\mathbf{y}_k \to \mathbf{y}$ in E, we have $f(\mathbf{y}_k) \to f(\mathbf{y})$.

Theorem. Let $(V, \|\cdot\|)$ and $(V', \|\cdot\|')$ be normed spaces, and K a compact subset of V, and $f : V \to V'$ a continuous function. Then

(i) $f(K)$ is compact in V'

(ii) $f(K)$ is closed and bounded

(iii) If $V' = \mathbb{R}$, then the function attains its supremum and infimum, i.e. there is some $\mathbf{y}_1, \mathbf{y}_2 \in K$ such that

$$f(\mathbf{y}_1) = \sup\{f(\mathbf{y}) : \mathbf{y} \in K\}, \quad f(\mathbf{y}_2) = \inf\{f(\mathbf{y}) : \mathbf{y} \in K\}.$$

Lemma. Let V be an n-dimensional vector space with a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. Then for any $\mathbf{x} \in V$, write $\mathbf{x} = \sum_{j=1}^n x_j \mathbf{v}_j$, with $x_j \in \mathbb{R}$. We define the Euclidean norm by

$$\|\mathbf{x}\|_2 = \left(\sum x_j^2\right)^{\frac{1}{2}}.$$

Then this is a norm, and $S = \{\mathbf{x} \in V : \|\mathbf{x}\|_2 = 1\}$ is compact in $(V, \|\cdot\|_2)$.

Theorem. Any two norms on a finite dimensional vector space are Lipschitz equivalent.

Corollary. Let $(V, \|\cdot\|)$ be a finite-dimensional normed space.

(i) The Bolzano-Weierstrass theorem holds for V, i.e. any bounded sequence sequence in V has a convergent subsequence.

(ii) A subset of V is compact if and only if it is closed and bounded.

Corollary. Any finite-dimensional normed vector space $(V, \|\cdot\|)$ is complete.

5 Metric spaces

5.1 Preliminary definitions

Proposition. The limit of a convergent sequence is unique.

5.2 Topology of metric spaces

Proposition. Let (X, d) be a metric space. Then $x_k \to x$ if and only if for every neighbourhood V of x, there exists some K such that $x_k \in V$ for all $k \geq K$. Hence convergence is a topological notion.

Theorem. Let (X, d) be a metric space. Then

(i) The union of *any* collection of open sets is open

(ii) The intersection of finitely many open sets is open.

(iii) $\emptyset$ and X are open.

Proposition. A subset is closed if and only if its complement is open.

Theorem. Let (X, d) be a metric space. Then

(i) The intersection of *any* collection of closed sets is closed

(ii) The union of finitely many closed sets is closed.

(iii) $\emptyset$ and X are closed.

Proposition. Let (X, d) be a metric space and $x \in X$. Then the singleton $\{x\}$ is a closed subset, and hence any finite subset is closed.

5.3 Cauchy sequences and completeness

Proposition. Let (X, d) be a metric space. Then

(i) Any convergent sequence is Cauchy.

(ii) If a Cauchy sequence has a convergent subsequence, then the original sequence converges to the same limit.

Theorem. Let (X, d) be a metric space, $Y \subseteq X$ any subset. Then

(i) If $(Y, d|_{Y \times Y})$ is complete, then Y is closed in X.

(ii) If (X, d) is complete, then $(Y, d|_{Y \times Y})$ is complete if and only if it is closed.

5.4 Compactness

Theorem. All compact spaces are complete and bounded.

Theorem. (non-examinable) Let (X, d) be a metric space. Then X is compact if and only if X is complete and totally bounded.

5.5 Continuous functions

Theorem. Let (X, d) be a compact metric space, and (X', d') is any metric space. If $f : X \to X'$ be continuous, then f is uniformly continuous.

Theorem. Let (X, d) and (X', d') be metric spaces, and $f : X \to X'$. Then the following are equivalent:

(i) f is continuous at y.

(ii) $f(x_k) \to f(y)$ for every sequence (x_k) in X with $x_k \to y$.

(iii) For every neighbourhood V of $f(y)$, there is a neighbourhood U of y such that $U \subseteq f^{-1}(V)$.

Corollary. A function $f : (X, d) \to (X', d')$ is continuous if $f^{-1}(V)$ is open in X whenever V is open in X'.

5.6 The contraction mapping theorem

Theorem (Contraction mapping theorem)**.** Let X be a (non-empty) complete metric space, and if $f : X \to X$ is a contraction, then f has a *unique* fixed point, i.e. there is a unique x such that $f(x) = x$.

Moreover, if $f : X \to X$ is a function such that $f^{(m)} : X \to X$ (i.e. f composed with itself m times) is a contraction for some m, then f has a unique fixed point.

Theorem (Picard-Lindelöf existence theorem)**.** Let $\mathbf{x}_0 \in \mathbb{R}^n$, $R > 0$, $a < b$, $t_0 \in [a, b]$. Let $\mathbf{F} : [a, b] \times \overline{B_R(\mathbf{x}_0)} \to \mathbb{R}^n$ be a continuous function satisfying

$$\|\mathbf{F}(t, \mathbf{x}) - \mathbf{F}(t, \mathbf{y})\|_2 \leq \kappa \|\mathbf{x} - \mathbf{y}\|_2$$

for some fixed $\kappa > 0$ and all $t \in [a, b], \mathbf{x} \in \overline{B_R(\mathbf{x}_0)}$. In other words, $F(t, \cdot\,) : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz on $\overline{B_R(\mathbf{x}_0)}$ with the same Lipschitz constant for every t. Then

(i) There exists an $\varepsilon > 0$ and a unique differentiable function $\mathbf{f} : [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a, b] \to \mathbb{R}^n$ such that

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} = \mathbf{F}(t, \mathbf{f}(t)) \tag{$*$}$$

and $\mathbf{f}(t_0) = \mathbf{x}_0$.

(ii) If

$$\sup_{[a,b] \times \overline{B_R(\mathbf{x}_0)}} \|\mathbf{F}\|_2 \leq \frac{R}{b - a},$$

then there exists a unique differential function $\mathbf{f} : [a, b] \to \mathbb{R}^n$ that satisfies the differential equation and boundary conditions above.

6 Differentiation from $\mathbb{R}^m$ to $\mathbb{R}^n$

6.1 Differentiation from $\mathbb{R}^m$ to $\mathbb{R}^n$

Proposition (Uniqueness of derivative)**.** Derivatives are unique.

Proposition. Let $U \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in U$.

(i) If $\mathbf{f} : U \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$, then $\mathbf{f}$ is continuous at $\mathbf{a}$.

(ii) If we write $\mathbf{f} = (f_1, f_2, \cdots, f_m) : U \to \mathbb{R}^m$, where each $f_i : U \to \mathbb{R}$, then $\mathbf{f}$ is differentiable at $\mathbf{a}$ if and only if each f_j is differentiable at $\mathbf{a}$ for each j.

(iii) If $f, g : U \to \mathbb{R}^m$ are both differentiable at $\mathbf{a}$, then $\lambda\mathbf{f} + \mu\mathbf{g}$ is differentiable at $\mathbf{a}$ with
$$D(\lambda\mathbf{f} + \mu\mathbf{g})(\mathbf{a}) = \lambda D\mathbf{f}(\mathbf{a}) + \mu D\mathbf{g}(\mathbf{a}).$$

(iv) If $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then A is differentiable for any $\mathbf{a} \in \mathbb{R}^n$ with
$$DA(\mathbf{a}) = A.$$

(v) If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then the directional derivative $D_{\mathbf{u}}\mathbf{f}(\mathbf{a})$ exists for all $\mathbf{u} \in \mathbb{R}^n$, and in fact
$$D_{\mathbf{u}}\mathbf{f}(\mathbf{a}) = D\mathbf{f}(\mathbf{a})\mathbf{u}.$$

(vi) If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then all partial derivatives $D_j f_i(\mathbf{a})$ exist for $j = 1, \cdots, n$; $i = 1, \cdots, m$, and are given by
$$D_j f_i(\mathbf{a}) = Df_i(\mathbf{a})\mathbf{e}_j.$$

(vii) If $A = (A_{ij})$ be the matrix representing $D\mathbf{f}(\mathbf{a})$ with respect to the standard basis for $\mathbb{R}^n$ and $\mathbb{R}^m$, i.e. for any $\mathbf{h} \in \mathbb{R}^n$,
$$D\mathbf{f}(\mathbf{a})\mathbf{h} = A\mathbf{h}.$$
Then A is given by
$$A_{ij} = \langle D\mathbf{f}(\mathbf{a})\mathbf{e}_j, \mathbf{b}_i\rangle = D_j\mathbf{f}_i(a).$$
where $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$, and $\{\mathbf{b}_1, \cdots, \mathbf{b}_m\}$ is the standard basis for $\mathbb{R}^m$.

Theorem. Let $U \subseteq \mathbb{R}^n$ be open, $\mathbf{f} : U \to \mathbb{R}^m$. Let $\mathbf{a} \in U$. Suppose there exists some open ball $B_r(\mathbf{a}) \subseteq U$ such that

(i) $D_j\mathbf{f}_i(\mathbf{x})$ exists for every $\mathbf{x} \in B_r(\mathbf{a})$ and $1 \leq i \leq m$, $1 \leq j \leq n$

(ii) $D_j\mathbf{f}_i$ are continuous at $\mathbf{a}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Then $\mathbf{f}$ is differentiable at $\mathbf{a}$.

6.2 The operator norm

Proposition.

(i) $\|A\| < \infty$ for all $A \in \mathcal{L}$.

(ii) $\|\cdot\|$ is indeed a norm on $\mathcal{L}$.

(iii)
$$\|A\| = \sup_{\mathbb{R}^n \setminus \{0\}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}.$$

(iv) $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.

(v) Let $A \in L(\mathbb{R}^n; \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m; \mathbb{R}^p)$. Then $BA = B \circ A \in L(\mathbb{R}^n; \mathbb{R}^p)$ and
$$\|BA\| \leq \|B\|\|A\|.$$

Proposition.

(i) If $A \in L(\mathbb{R}, \mathbb{R}^m)$, then A can be written as $Ax = x\mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^m$. Moreover, $\|A\| = \|\mathbf{a}\|$, where the second norm is the Euclidean norm in $\mathbb{R}^n$

(ii) If $A \in L(\mathbb{R}^n, \mathbb{R})$, then $A\mathbf{x} = \mathbf{x} \cdot \mathbf{a}$ for some fixed $\mathbf{a} \in \mathbb{R}^n$. Again, $\|A\| = \|\mathbf{a}\|$.

Theorem (Chain rule)**.** Let $U \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in U$, $\mathbf{f} : U \to \mathbb{R}^m$ differentiable at $\mathbf{a}$. Moreover, $V \subseteq \mathbb{R}^m$ is open with $\mathbf{f}(U) \subseteq V$ and $\mathbf{g} : V \to \mathbb{R}^p$ is differentiable at $\mathbf{f}(\mathbf{a})$. Then $\mathbf{g} \circ \mathbf{f} : U \to \mathbb{R}^p$ is differentiable at $\mathbf{a}$, with derivative
$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{a}) = D\mathbf{g}(\mathbf{f}(\mathbf{a}))\ D\mathbf{f}(\mathbf{a}).$$

6.3 Mean value inequalities

Theorem. Let $\mathbf{f} : [a, b] \to \mathbb{R}^m$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose we can find some M such that for all $t \in (a, b)$, we have $\|D\mathbf{f}(t)\| \leq M$. Then
$$\|\mathbf{f}(b) - \mathbf{f}(a)\| \leq M(b - a).$$

Theorem (Mean value inequality)**.** Let $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{f} : B_r(\mathbf{a}) \to \mathbb{R}^m$ be differentiable on $B_r(\mathbf{a})$ with $\|D\mathbf{f}(\mathbf{x})\| \leq M$ for all $\mathbf{x} \in B_r(\mathbf{a})$. Then
$$\|\mathbf{f}(\mathbf{b}_1) - f(\mathbf{b}_2)\| \leq M\|\mathbf{b}_1 - \mathbf{b}_2\|$$
for any $\mathbf{b}_1, \mathbf{b}_2 \in B_r(\mathbf{a})$.

Corollary. Let $\mathbf{f} : B_r(\mathbf{a}) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ have $D\mathbf{f}(\mathbf{x}) = 0$ for all $\mathbf{x} \in B_r(\mathbf{a})$. Then $\mathbf{f}$ is constant.

Theorem. Let $U \subseteq \mathbb{R}^n$ be open and path-connected. Then for any differentiable $\mathbf{f} : U \to \mathbb{R}^m$, if $D\mathbf{f}(\mathbf{x}) = 0$ for all $\mathbf{x} \in U$, then $\mathbf{f}$ is constant on U.

6.4 Inverse function theorem

Proposition. Let $U \subseteq \mathbb{R}^n$ be open. Then $\mathbf{f} = (f_1, \cdots, f_n) : U \to \mathbb{R}^n$ is C^1 on U if and only if the partial derivatives $D_j f_i(\mathbf{x})$ exists for all $\mathbf{x} \in U$, $1 \leq i \leq n$, $1 \leq j \leq n$, and $D_j f_i : U \to \mathbb{R}$ are continuous.

Theorem (Inverse function theorem)**.** Let $U \subseteq \mathbb{R}^n$ be open, and $\mathbf{f} : U \to \mathbb{R}^m$ be a C^1 map. Let $\mathbf{a} \in U$, and suppose that $D\mathbf{f}(\mathbf{a})$ is invertible as a linear map $\mathbb{R}^n \to \mathbb{R}^n$. Then there exists open sets $V, W \subseteq \mathbb{R}^n$ with $\mathbf{a} \in V$, $\mathbf{f}(\mathbf{a}) \in W$, $V \subseteq U$ such that
$$\mathbf{f}|_V : V \to W$$
is a bijection. Moreover, the inverse map $\mathbf{f}|_V^{-1} : W \to V$ is also C^1.

6.5 2nd order derivatives

Theorem (Symmetry of mixed partials)**.** Let $U \subseteq \mathbb{R}^n$ be open, $\mathbf{f} : U \to \mathbb{R}^m$, $\mathbf{a} \in U$, and $\rho > 0$ such that $B_\rho(\mathbf{a}) \subseteq U$.

Let $i, j \in \{1, \cdots, n\}$ be fixed and suppose that $D_i D_j \mathbf{f}(\mathbf{x})$ and $D_j D_i \mathbf{f}(\mathbf{x})$ exist for all $\mathbf{x} \in B_\rho(\mathbf{a})$ and are continuous at $\mathbf{a}$. Then in fact
$$D_i D_j \mathbf{f}(\mathbf{a}) = D_j D_i \mathbf{f}(\mathbf{a}).$$

Proposition. If $f : U \to \mathbb{R}^m$ is differentiable in U such that $D_i D_j \mathbf{f}(\mathbf{x})$ exists in a neighbourhood of $\mathbf{a} \in U$ and are continuous at $\mathbf{a}$, then $D\mathbf{f}$ is differentiable at $\mathbf{a}$ and
$$D^2\mathbf{f}(\mathbf{a})(\mathbf{u}, \mathbf{v}) = \sum_j \sum_i D_i D_j \mathbf{f}(\mathbf{a}) u_i v_j.$$
is a symmetric bilinear form.

Theorem (Second-order Taylor's theorem)**.** Let $f : U \to \mathbb{R}$ be C^2, i.e. $D_i D_j f(\mathbf{x})$ are continuous for all $\mathbf{x} \in U$. Let $\mathbf{a} \in U$ and $B_r(\mathbf{a}) \subseteq U$. Then
$$f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + Df(\mathbf{a})\mathbf{h} + \frac{1}{2} D^2 f(\mathbf{h}, \mathbf{h}) + E(\mathbf{h}),$$
where $E(\mathbf{h}) = o(\|\mathbf{h}\|^2)$.