ANALYSIS II—EXAMPLES 1 Mich. 2015

Please email comments, corrections to: n.wickramasekera@dpmms.cam.ac.uk

1. Quickies: (a) If (f_n) is a sequence of real functions converging uniformly on $[0,1]$ to a function f, and if f_n is continuous at $x_n \in [0,1]$ with $x_n \to x$, does it follow that f is continuous at x?

(b) If (f_n) is a sequence of continuous functions converging pointwise on $[-1,1]$ to a continuous function f, and if the convergence is uniform on $[-r,r]$ for every $r \in (0,1)$, does it follow that the convergence is uniform on $[-1,1]$?

(c) If (f_n) is a sequence of functions converging uniformly on $[0,1]$ to a function f, and if each f_n is continuous except at countably many points, does it follow that there exists a point at which f is continuous?

(d) If (f_n) is a sequence of differentiable functions on $[0,1]$ converging uniformly to a function f on $[0,1]$, does it follow that there exists a point at which f is differentiable?

2. Which of the following sequences (f_n) of functions converge uniformly on the set X?

(a) $f_n(x) = x^n$ on $X = (0,1)$; (b) $f_n(x) = x^n$ on $X = (0, \frac{1}{2})$; (c) $f_n(x) = xe^{-nx}$ on $X = [0,\infty)$;

(d) $f_n(x) = e^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.

3. Let (f_n) and (g_n) be sequences of real-valued functions on a subset of $\mathbb{R}$ converging uniformly to f and g respectively. Show that the pointwise sum $f_n + g_n$ converges uniformly to $f+g$. On the other hand, show that the pointwise product $f_n g_n$ need not converge uniformly to fg, but that if both f and g are bounded then $f_n g_n$ does converge uniformly to fg. What if f is bounded but g is not?

4. Let (f_n) be a sequence of bounded, real-valued functions on a subset of $\mathbb{R}$ converging uniformly to a function f. Show that f must be bounded. Give an example of a sequence (g_n) of bounded, real-valued functions on $[-1,1]$ converging pointwise to a function g which is not bounded.

5. Let (f_n) be a sequence of real-valued continuous functions on a closed, bounded interval $[a,b]$, and suppose that f_n converges pointwise to a continuous function f. Show that if $f_n \to f$ uniformly and (x_m) is a sequence of points in $[a,b]$ with $x_m \to x$ then $f_n(x_n) \to f(x)$. On the other hand, show that if f_n does not converge uniformly to f then we can find a convergent sequence $x_m \to x$ in $[a,b]$ such that $f_n(x_n) \not\to f(x)$.

6. Let (f_n) be a sequence of real-valued functions on $[0,1]$ converging uniformly to a function f.

(a) If $\mathcal{D}_n$ is the set of discontinuities of f_n and $\mathcal{D}$ is the set of discontinuities of f, show that $\mathcal{D} \subseteq \cup_{n=1}^{\infty} \cap_{j=n}^{\infty} \mathcal{D}_j$.

(b) Suppose that for some finite k, each f_n is discontinuous at most at k points. What can you say about the set of discontinuities of f?

7. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.

(a) Define a sequence (f_n) of functions on $[-\pi,\pi]$ by $f_n(x) = \sum_{m=1}^{n} a_m \sin mx$. Show that each f_n is differentiable with $f_n'(x) = \sum_{m=1}^{n} m a_m \cos mx$.

(b) Show that $f(x) = \sum_{m=1}^{\infty} a_m \sin mx$ defines a continuous function on $[-\pi,\pi]$, but that the series $\sum_{m=1}^{\infty} m a_m \cos mx$ need not converge.

8. Show that, for any $x \in X = \mathbb{R} - \{1,2,3,\dots\}$, the series $\sum_{m=1}^{\infty} (x-m)^{-2}$ converges. Define $f \colon X \to \mathbb{R}$ by $f(x) = \sum_{m=1}^{\infty} (x-m)^{-2}$, and for $n = 1, 2, 3, \dots$, define $f_n \colon X \to \mathbb{R}$ by $f_n(x) = \sum_{m=1}^{n} (x-m)^{-2}$. Does the sequence (f_n) converge uniformly to f on X? Is f continuous?

9. Let a_n be real numbers such that $\sum_{n=0}^{\infty} a_n$ converges.

(a) Show that $\sum_{n=1}^{\infty} a_n x^n$ converges for $x \in (-1,1)$. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, show that f is differentiable on $(-1,1)$.

(b)* Show that f extends to $(-1,1]$ as a continuous function with $f(1) = \sum_{n=0}^{\infty} a_n$. (Hint: start by showing that $f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n$ for $|x| < 1$, where $s_n = \sum_{j=0}^{n} a_j$.) Show that, for each $r \in (-1,1)$, the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[r,1]$. Must the one-sided derivative $f'(1)$ exist?

10. Is there a real power series with radius of convergence 1 that converges uniformly on $(-1,1)$?

11. Which of the following functions $f\colon [0,\infty) \to \mathbb{R}$ are (a) uniformly continuous; (b) bounded?

(i)$f(x) = \sin x^2$; (ii) $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$; (iii) $f(x) = (\sin x^3)/(x+1)$.

12. Show that if (f_n) is a sequence of uniformly continuous, real-valued functions on $\mathbb{R}$, and if $f_n \to f$ uniformly, then f is uniformly continuous. Give an example of a sequence of uniformly continuous, real-valued functions (f_n) on $\mathbb{R}$ such that f_n converges pointwise to a function f which is continuous but not uniformly continuous.

13. Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is continuous, and that $f(x)$ tends to a (finite) limit as $x \to \infty$. Must f be uniformly continuous on $[0,\infty)$? Give a proof or counterexample as appropriate.

14. Let f be a differentiable, real-valued function on $\mathbb{R}$, and suppose that f' is bounded. Show that f is uniformly continuous. Let $g\colon [-1,1] \to \mathbb{R}$ be the function defined by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $g(0) = 0$. Show that g is differentiable, but that its derivative is unbounded. Is g uniformly continuous?

15. Let f be a bounded real-valued Riemann integrable functions on $[0,1]$.

(a) Must there exist a sequence (f_n) of continuous functions on $[0,1]$ such that $f_n \to f$ uniformly on $[0,1]$?

(b)* Must there exist a sequence (f_n) of continuous functions on $[0,1]$ such that $\int_0^1 |f_n(x) - f(x)|\, dx \to 0$?

(c)* Must there exist a sequence (p_n) of polynomials such that $\int_0^1 |p_n(x) - f(x)|\, dx \to 0$?

16*. Define $\varphi(x) = |x|$ for $x \in [-1,1]$ and extend the definition of $\varphi(x)$ to all real x by requiring that $\varphi(x+2) = \varphi(x)$.

(i) Show that $|\varphi(s) - \varphi(t)| \leq |s - t|$ for all s and t.

(ii) Define $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$. Prove that f is well-defined and continuous.

(iii) Fix a real number x and positive integer m. Put $\delta_m = \pm\frac{1}{2}4^{-m}$, where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x+\delta_m)$. Prove that

$$\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| \geq \frac{1}{2}(3^m + 1).$$

Conclude that f is not differentiable at x. Hence there exists a real continuous function on the real line which is nowhere differentiable.

ANALYSIS II—EXAMPLES 2 Mich. 2015

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1. Quickies: (a) Describe all continuous functions $f : [0,1] \to \mathbb{R}^n$ satisfying $\| \int_0^1 f \| = \int_0^1 \|f\|$.

(b) Show that two norms $\| \cdot \|$, $\| \cdot \|'$ on a vector space V are Lipschitz equivalent if and only if there exist numbers $r, R > 0$ such that $B_r \subseteq B'_1 \subseteq B_R$, where for $\rho > 0$, $B_\rho = \{x \in V : \|x\| < \rho\}$ and $B'_\rho = \{x \in V : \|x\|' < \rho\}$.

(c) If $(V, \| \cdot \|)$ is a normed space and $\varphi : V \to \mathbb{R}$ is a linear functional, show that $\| \cdot \| + |\varphi(\cdot)|$ defines a norm on V, and that this norm is not Lipschitz equivalent to $\| \cdot \|$ if φ is not continuous.

(d) If a Cauchy sequence (x_n) in a normed space has a subsequence converging to an element x, show that the whole sequence (x_n) converges to x.

2. Let $(x^{(m)})$ and $(y^{(m)})$ be sequences in $\mathbb{R}^n$ converging to x and y respectively. Show that $x^{(m)} \cdot y^{(m)}$ converges to $x \cdot y$. Deduce that if $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are continuous at $x \in \mathbb{R}^n$, then so is the pointwise scalar product $f \cdot g : \mathbb{R}^n \to \mathbb{R}$.

3. (a) Show that $\|f\|_1 = \int_0^1 |f|$ defines a norm on the vector space $C\big([0,1]\big)$. Is it Lipschitz equivalent to the uniform norm? Is $C\big([0,1]\big)$ with norm $\| \cdot \|_1$ complete?

(b) Let $R\big([0,1]\big)$ denote the vector space of all bounded Riemann integrable functions on $[0,1]$. Does $\|f\|_1 = \int_0^1 |f|$ define a norm on $R\big([0,1]\big)$? If so, is $R\big([0,1]\big)$ complete with this norm? What if we replace $\| \cdot \|_1$ with $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$?

4. (a) Let $C^1([0,1])$ be the vector space of real continuous functions on $[0,1]$ with continuous first derivatives. Define functions $\alpha, \beta, \gamma, \delta : C^1([0,1]) \to \mathbb{R}$ by $\alpha(f) = \sup_{x \in [0,1]} |f(x)| + \sup_{x \in [0,1]} |f'(x)|$; $\beta(f) = \sup_{x \in [0,1]} (|f(x)| + |f'(x)|)$; $\gamma(f) = \sup_{x \in [0,1]} |f(x)|$; $\delta(f) = \sup_{x \in [0,1]} |f'(x)|$. Which of these define norms on $C^1([0,1])$? Out of those that define norms, which pairs are Lipschitz equivalent?

(b) Let $C^1_c([0,1])$ be the set of functions $f \in C^1([0,1])$ such that $f(x) = 0$ for x in some neighborhood of the end points 0 and 1. Verify that $C^1_c([0,1])$ is a vector space. How would your answers in (a) change if we replace $C^1([0,1])$ by $C^1_c([0,1])$?

5. Which of the following subsets of $\mathbb{R}^2$ with the Euclidean norm are open? Which are closed? (And why?)

(i) $\{(x,0) : 0 \le x \le 1\}$;
(ii) $\{(x,0) : 0 < x < 1\}$;
(iii) $\{(x,y) : y \ne 0\}$;
(iv) $\{(x,y) : x \in \mathbb{Q} \quad \text{or} \quad y \in \mathbb{Q}\}$;
(v) $\{(x,y) : y = nx \quad \text{for some} \quad n \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$;
(vi) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

6. Is the set $\{f : f(1/2) = 0\}$ closed in the space $C\big([0,1]\big)$ with the uniform norm? What about the set $\{f : \int_0^1 f = 0\}$? In each case, does the answer change if we replace the uniform norm with the norm $\| \cdot \|_1$ defined in Q3?

7. Which of the following functions f are continuous?

(i) The linear map $f \colon \ell^\infty \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^\infty x_n/n^2$;
(ii) The identity map from the space $C\big([0,1]\big)$ with the uniform norm $\| \cdot \|$ to the space $C\big([0,1]\big)$ with the norm $\| \cdot \|_1$ defined in Q3;
(iii) The identity map from $C\big([0,1]\big)$ with the norm $\| \cdot \|_1$ to $C\big([0,1]\big)$ with the uniform norm $\| \cdot \|$;
(iv) The linear map $f \colon \ell^0 \to \mathbb{R}$ defined by $f(x) = \sum_{i=1}^\infty x_i$, where ℓ^0 has norm $\| \cdot \|_\infty$. (ℓ^0 is the space of real sequences (x_k) such that $x_k = 0$ for all but a finite number of k.)

8. Is it possible to find uncountably many norms on $C([0,1])$ such that no two are Lipschitz equivalent?

9. Let ℓ^1 denote the set of real sequences (x_n) such that $\sum_{n=1}^\infty |x_n|$ is convergent. Show that, with addition and scalar multiplication defined termwise, ℓ^1 is a vector space. Define $\| \cdot \|_1 \colon \ell^1 \to \mathbb{R}$ by $\|x\|_1 = \sum_{n=1}^\infty |x_n|$. Show that $\| \cdot \|_1$ is a norm on ℓ^1, and that $(\ell^1, \| \cdot \|_1)$ is complete.

$10^\star$. Let $(V, \|\cdot\|)$ be a normed space. Show that V is complete if and only if V has the property that for every sequence (x_n) in V with $\sum_{j=1}^\infty \|x_n\|$ convergent, the series $\sum_{n=1}^\infty x_n$ is convergent. (Thus V is complete if and only if every absolutely convergent series in V is convergent.) [Hint: If (x_n) is Cauchy, then there is a subsequence (x_{n_j}) such that $\sum_j \|x_{n_{j+1}} - x_{n_j}\| < \infty$.]

11. Let V be a normed space in which every bounded sequence has a convergent subsequence. (a) Show that this property of V is equivalent to the sequential compactness of the unit sphere $S = \{x \in V : \|x\| = 1\}$. (b) Show that V must be complete. (c)* Show further that V must be finite-dimensional.
[Hint for (c): Start by showing that for every finite-dimensional subspace V_0 of V, there exists $x \in V$ with $\|x + y\| > \|x\|/2$ for each $y \in V_0$.]

12. Let $(x^{(n)})_{n\geq 1}$ be a bounded sequence in ℓ^∞. Show that there is a subsequence $(x^{(n_j)})_{j\geq 1}$ which converges in every coordinate; that is to say, the sequence $(x_i^{(n_j)})_{j\geq 1}$ of real numbers converges for each i. Why does this not show that every bounded sequence in ℓ^∞ has a convergent subsequence?

13. (a) Let $(V, \|\cdot\|)$ be a complete normed space, and let W be a subspace of V. Show that $(W, \|\cdot\|)$ is complete if and only if W is closed in V.

(b) Which of the following vector spaces of functions, taken with the uniform norm, are complete?

(i) The space $C_b(\mathbb{R})$ of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$.
(ii) The space $C_0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$.
(iii) The space $C_c(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $|x|$ sufficiently large.

$14^\star$. Let $\mathcal{P}$ be the vector space of real polynomials on the unit interval $[0,1]$. Show that for any infinite set $I \subseteq [0,1]$, $\|p\|_I = \sup_I |p|$ defines a norm on $\mathcal{P}$. Use this fact to produce an example of a vector space, a sequence in it and two different norms on it such that the sequence converges to different elements in the space with respect to the different norms. (Hint: the Weierstrass approximation theorem may be helpful).

Is it possible to find such a sequence in one of the spaces ℓ^1 or ℓ^2 equipped with two norms, when possible, chosen from the standard norms on the spaces ℓ^1, ℓ^2, ℓ^∞? What about in the space $C([0,1])$ equipped with two norms chosen from the L^1, L^2, L^∞ norms?

Supplement: A proof of Lebesgue's theorem on the Riemann integral. Let $f : [a,b] \to \mathbb{R}$ be bounded. Recall that Lebesgue's theorem says that f is Riemann integrable on $[a,b]$ if and only if the set $\mathcal{D}_f$ of points in $[a,b]$ where f is discontinuous has Lebesgue measure zero. (By definition, a set $\mathcal{D} \subset \mathbb{R}$ has Lebesgue measure zero if for every $\epsilon > 0$, there is a countable collection of open intervals $I_j = (a_j, b_j)$ such that $\mathcal{D} \subset \cup_{j=1}^\infty I_j$ and $\sum_{j=1}^\infty |I_j| < \epsilon$, where $|I_j| = b_j - a_j$.) As an optional exercise, prove this theorem by completing the outline below. We shall use the notation as in lectures, so $U(P,f)$, $L(P,f)$ denote the upper and lower sums for f relative to a partition P of $[a,b]$.

(a) Show that $y \in \mathcal{D}_f \cap (a,b)$ (i.e. y is an interior discontinuity) if and only if there exists $\epsilon = \epsilon_y > 0$ such that $\sup_I f - \inf_I f > \epsilon$ for every open interval $I \subset [a,b]$ with $y \in I$. Hence $\mathcal{D}_f \cap (a,b) = \cup_{j=1}^\infty E_j$, where $E_j = \{y \in (a,b) : \sup_I f - \inf_I f > j^{-1} \text{ for every open interval } I \text{ with } y \in I\}$.

(b) Suppose that f is Riemann integrable. It suffices to show that E_j has Lebesgue measure zero for each j (Why?). Fix j, let $\epsilon > 0$ and choose a partition $P = \{a = a_0 < a_1 < \ldots < a_n = b\}$ such that $U(P,f) - L(P,f) < j^{-1}\epsilon$. Let $K = \{k : E_j \cap (a_k, a_{k+1}) \neq \emptyset\}$. Then $E_j \setminus \{a_0, a_1, \ldots, a_n\} \subset \cup_{k\in K}(a_k, a_{k+1})$. Show that $\sum_{k\in K}(a_{k+1} - a_k) < \epsilon$. Deduce that E_j has Lebesgue measure zero.

(c) Now suppose that $\mathcal{D}_f$ has Lebesgue measure zero. Let $\epsilon > 0$, and choose open intervals $I_j \subset \mathbb{R}$, $j = 1, 2, \ldots$, with $\mathcal{D}_f \subset \cup_{j=1}^\infty I_j$ and $\sum_{j=1}^\infty |I_j| < \epsilon$. Let $F = [a,b] \setminus \cup_{j=1}^\infty I_j$. Show that there exists $\delta > 0$ such that the following holds: $x \in F$, $y \in [a,b]$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. [This is a strengthening of the theorem we proved in lecture that says that a continuous function on a closed, bounded interval (or more generally on a compact metric space) is uniformly continuous, but the same contradiction argument we used in fact works here.] Let $P = \{a = a_0 < a_1 < a_2 \ldots < a_n = b\}$ be any partition of $[a,b]$ such that $a_{j+1} - a_j < \delta$, and let $J = \{j : [a_j, a_{j+1}] \cap F \neq \emptyset\}$. Show that $\sup_{[a_j,a_{j+1}]} f - \inf_{[a_j,a_{j+1}]} f < 2\epsilon$ for each $j \in J$, and that $\cup_{j\notin J}(a_j, a_{j+1}) \subset \cup_{j=1}^\infty I_j$. Conclude that $U(P,f) - L(P,f) < 2(b - a + \sup_{[a,b]} |f|)\epsilon$, and hence that f is Riemann integrable on $[a,b]$.

ANALYSIS II—EXAMPLES 3 Mich. 2015

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1. Quickies: (a) Use the equivalence of norms on a finite dimensional vector space to show that for each n, there is a constant C such that the following holds: for every polynomial p of degree $\leq n$ there is $x_0 \in [0, 1/n]$ such that $|p(x)| \leq C|p(x_0)|$ for every $x \in [0, 1]$.

(b) If (X, d) is a metric space and A is a non-empty subset of X, show that the distance from $x \in X$ to A defined by $\rho(x) = \inf_{y \in A} d(x, y)$ is a Lipschitz function on X with Lipschitz constant equal to 1.

(c) If every closed, bounded subset of a metric space X is compact, must X be complete?

(d) If every closed proper subset of a metric space X is complete relative to the induced metric, must X be complete?

(e) If (x_n), (y_n) are Cauchy sequences in a metric space (X, d), show that $(d(x_n, y_n))$ is convergent (in $\mathbb{R}$).

2. (a) Is the set $(1, 2]$ an open subset of the metric space $\mathbb{R}$ with the metric $d(x, y) = |x - y|$? Is it closed? What if we replace the metric space $\mathbb{R}$ with the space $[0, 2]$, the space$(1, 3)$ or the space $(1, 2]$, in each case with the metric d?

(b) Let X be a set equipped with the discrete metric, and Y any metric space. Describe all open subsets of X, closed subsets of X, sequentially compact subsets of X, Cauchy sequences in X, continuous functions $X \to Y$ and continuous functions $Y \to X$.

3. For each of the following sets X, determine whether or not the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $B_\varepsilon(x)$ for $x \in X$ and $\varepsilon > 0$ small.

(i) $X = \mathbb{R}^n$; $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$.

(ii) $X = \mathbb{Z}$; $d(x, x) = 0$, and, for $x \neq y$, $d(x, y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.

(iii) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f, f) = 0$, and, for $f \neq g$, $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(iv) $X = \mathbb{C}$; $d(z, w) = |z - w|$ if z and w lie on the same line through the origin, $d(z, w) = |z| + |w|$ otherwise.

4. Let (X, d) be a metric space.

(a) Show that the union of any collection of open subsets of X must be open (regardless of whether the collection is finite, countable or uncountable), and that the intersection of any finite collection of open subsets is again open. Formulate and prove similar properties about the closed subsets of X.

(b) Let E be a subset of X. Show that there is a unique largest open subset E^o of X contained in E, i.e. a unique open subset E^o of X such that that $E^o \subseteq E$ and if G is any open subset of X with $G \subseteq E$ then $G \subseteq E^o$. E^o is called the *interior* of E in X. Show also that there is a unique smallest closed subset $\overline{E}$ of X containing E, i.e. a unique closed subset $\overline{E}$ of X with $E \subseteq \overline{E}$ and if F is any closed subset of X with $E \subseteq F$ then $\overline{E} \subseteq F$. $\overline{E}$ is called the *closure* of E in X.

(c) Show that

$$E^o = \{x \in X \,:\, B_\epsilon(x) \subset E \quad \text{for} \quad \text{some} \quad \epsilon > 0\}$$

and that

$$\overline{E} = \{x \in X \,:\, x_n \to x \quad \text{for} \quad \text{some} \quad \text{sequence} \quad (x_n) \quad \text{in} \quad E\}.$$

5. Let V be a normed space, $x \in V$ and $r > 0$. Prove that the closure of the open ball $B_r(x)$ is the closed ball $D_r(x) = \{y \in V : \|x - y\| \leq r\}$. Give an example to show that, in a general metric space (X, d), the closure of the open ball $B_r(x)$ need not be the closed ball $D_r(x) = \{y \in X : d(x, y) \leq r\}$.

6. In lectures we proved that if E is a closed, bounded subset of $\mathbb{R}^n$ with the Euclidean metric, then any continuous function on E has bounded image. Prove the converse: if E is a subset of $\mathbb{R}^n$ with the Euclidean metric and if every continuous function $f : E \to \mathbb{R}$ has bounded image, then E is closed and bounded.

7. Each of the following properties/notions makes sense for an arbitrary metric spaces X. Which are topological (i.e. dependent only on the collection of open subsets of X and not on the metric generating the open subsets)? Justify your answers.

(i) boundedness of a subset of X.

(ii) closed-ness of a subset of X.

(iii) notion that a subset of X is closed *and* bounded.

(iv) total boundedness of X; that is, the property that for every $\epsilon > 0$, there is a finite set $F \subset X$ such that the union of open balls with centres in F and radius ϵ is X.

(v) completeness of X.

(vi) total boundedness *and* completeness of X.

8. Use the Contraction Mapping Theorem to show that the equation $\cos x = x$ has a unique real solution. Find this solution to some reasonable accuracy using a calculator (remember to work in radians!), and justify the claimed accuracy of your approximation.

9. Let $I = [0, R]$ be an interval and let $C(I)$ be the space of continuous functions on I. Show that, for any $\alpha \in \mathbb{R}$, we may define a norm by $\|f\|_\alpha = \sup_{x\in I} |f(x)e^{-\alpha x}|$, and that the norm $\|\cdot\|_\alpha$ is Lipschitz equivalent to the uniform norm $\|f\| = \sup_{x\in I} |f(x)|$.
Now suppose that $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, and Lipschitz in the second variable. Consider the map T from $C(I)$ to itself sending f to $y_0 + \int_0^x \phi(t, f(t))dt$. Give an example to show that T need not be a contraction under the uniform norm. Show, however, that T is a contraction under the norm $\|\cdot\|_\alpha$ for some α, and hence deduce that the differential equation $f'(x) = \phi(x, f(x))$ has a unique solution on I satisfying $f(0) = y_0$.

10. Let (X, d) be a non-empty complete metric space. Suppose $f\colon X \to X$ is a contraction and $g\colon X \to X$ is a function which commutes with f, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that g has a fixed point. Must this fixed point be unique?

11. Give an example of a non-empty complete metric space (X, d) and a function $f\colon X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, but such that f has no fixed point. Suppose now that X is a non-empty closed bounded subset of $\mathbb{R}^n$ with the Euclidean metric. Show that in this case f must have a fixed point. If $g\colon X \to X$ satisfies $d(g(x), g(y)) \leq d(x, y)$ for all $x, y \in X$, must g have a fixed point?

12.* Show that it is not possible to obtain, starting from an arbitrary set $X \subseteq \mathbb{R}^n$ and repeatedly applying the operations $(\cdot)^o$ (interior) and $\overline{(\cdot)}$ (closure), more than seven distinct sets (including X itself). Give an example in $\mathbb{R}$ where seven sets are obtained.

13.* Let (X, d) be a non-empty complete metric space and let $f\colon X \to X$ be a function such that for each positive integer n we have
(i) if $d(x, y) < n + 1$ then $d(f(x), f(y)) < n$; and
(ii) if $d(x, y) < 1/n$ then $d(f(x), f(y)) < 1/(n + 1)$.
Must f have a fixed point?

14.* Let K be a closed bounded subset of $\mathbb{R}$ and $p \in K$. Construct a metric d on $K_1 = K \setminus \{p\}$ such that (K_1, d) is complete and the topology generated by d on K_1 is the same as the topology generated by the Euclidean metric on K_1.

15.* It is a consequence of the *Baire category theorem* (which you can learn about in the Linear Analysis course next year for example) that if f is the pointwise limit of a sequence of continuous functions $f_n : [a, b] \to \mathbb{R}$, then f has a point of continuity (and hence in fact a dense subset of $[a, b]$ of continuity points). Taking this fact for granted, and considering the family of functions $f_{n,m}(x) = (\cos n!\pi x)^{2m}$, $n, m \in \mathbb{N}$, show that pointwise convergence of continuous functions on an interval $[a, b]$ is not metrizable. That is to say, show that there is no metric d on the set of continuous functions $f : [a, b] \to \mathbb{R}$ such that pointwise convergence of sequences of functions in this set is equivalent to convergence with respect to d.

ANALYSIS II—EXAMPLES 4 Mich. 2015

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1. Quickies: (a) Let $F : [0,1] \times \mathbb{R}^m \to \mathbb{R}$ be continuous and $a = (a_0, \ldots, a_{m-1}) \in \mathbb{R}^m$. Suppose that F is uniformly Lipschitz in the $\mathbb{R}^m$ variables near a, i.e. for some constant K and an open subset U of $\mathbb{R}^m$ containing a, $|F(t,x) - F(t,y)| \leq K\|x-y\|$ for all $t \in [0,1]$, $x, y \in U$. Use the Picard–Lindelöf existence theorem for first order ODE systems to show that there is an $\epsilon > 0$ such that, writing $f^{(j)}$ for the jth derivative of f, the mth order initial value problem

$$f^{(m)}(t) = F(t, f(t), f^{(1)}(t), \ldots, f^{(m-1)}(t)) \quad \text{for} \quad t \in [0, \epsilon);$$

$$f^{(j)}(0) = a_j \quad \text{for} \quad 0 \leq j \leq m-1$$

has a unique C^m solution $f : [0,\epsilon) \to \mathbb{R}$ (see also Q2 below).

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If f is differentiable at $0 \in \mathbb{R}^2$, and if the partial derivatives of f exist in a neighborhood of 0, does it follow that one partial derivative is continuous at 0?

(c) Let $f\colon [a,b] \to \mathbb{R}^2$ be continuous, and differentiable on (a,b). Does it follow that there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$?

2. Let $x_0 \in \mathbb{R}^n$, $F : [a,b] \times \overline{B_R(x_0)} \to \mathbb{R}^n$ be continuous with $\sup_{[a,b]\times\overline{B_R(x_0)}} \|F\| \leq R(b-a)^{-1}$ and $\|F(t,x) - F(t,y)\| \leq K\|x-y\|$ for some K and all $t \in [a,b]$, $x, y \in \overline{B_R(x_0)}$. We showed in lecture that for each $t_0 \in [a,b]$, there is a unique $f \in C([a,b]; \overline{B_R(x_0)})$ solving the integral equation $f(t) = x_0 + \int_{t_0}^t F(s, f(s))\, ds$, $t \in [a,b]$. Assuming that F extends to all of $[a,b] \times \mathbb{R}^n$ as a continuous function, show that this f is in fact the unique function in $C([a,b];\mathbb{R}^n)$ solving the integral equation. (Hint: for $g \in C([a,b];\mathbb{R}^n)$ solving $g(t) = x_0 + \int_{t_0}^t F(s, g(s))\, ds$, $t \in [a,b]$, let $\Lambda^+ = \{t \in [t_0, b] : \|g(\sigma) - x_0\| \leq R \ \forall \sigma \in [t_0, t]\}$ and consider the possibility that $\sup \Lambda^+ < b$.)

3. (a) Let $f = (f_1, \ldots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$. Show that f is differentiable at $x \in \mathbb{R}^n$ iff each $f_i\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at x, and in this case, $Df(x)(h) = (Df_1(x)(h), \ldots, Df_m(x)(h))$ for each $h \in \mathbb{R}^n$.

(b) Define $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x,y,z) = (e^{x+y+z}, \cos x^2 y)$. Without making use of partial derivatives, show that f is everywhere differentiable and find $Df(a)$ at each $a \in \mathbb{R}^3$. Find all partial derivatives of f and hence, using appropriate results on partial derivatives, give an alternative proof of this result.

4. Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$. Show that f is differentiable except at 0, and that

$$Df(x)(h) = \frac{h}{\|x\|} - \frac{x(x \cdot h)}{\|x\|^3}.$$

Verify that $Df(x)(h)$ is orthogonal to x and explain geometrically why this is the case.

5. At which points of $\mathbb{R}^2$ is the function $f(x,y) = |x||y|$ differentiable? What about the function $g: \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x,y) = xy/\sqrt{x^2+y^2}$ if $(x,y) \neq (0,0)$, $g(0,0) = 0$?

6. Let f be a real-valued function on an open subset U of $\mathbb{R}^2$ such that that $f(\cdot, y)$ is continuous for each fixed $y \in U$ and $f(x, \cdot)$ is continuous for each fixed $x \in U$. Give an example to show that f need not be continuous on U. If additionally $f(\cdot, y)$ is Lipschitz for each $y \in U$ with Lipschitz constant independent of y, show that f is continuous on U. Deduce that if $D_1 f$ exists and is bounded on U and $f(x, \cdot)$ is continuous for each fixed $x \in U$, then f is continuous on U.

7. Let $f: \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}^2$. If $D_1 f$ exists in some open ball around a and is continuous at a, and if $D_2 f$ exists at a, show that f is differentiable at a.

8. (i) If $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^m \to \mathbb{R}^p$ are linear maps, show that $B \circ A : \mathbb{R}^n \to \mathbb{R}^p$ is linear and that $\|B \circ A\| \leq \|B\| \|A\|$ where $\|\cdot\|$ is the operator norm. (ii) If $A : \mathbb{R}^n \to \mathbb{R}$ is linear, show that there is $a \in \mathbb{R}^n$ such that $Ax = a \cdot x$ for all $x \in \mathbb{R}^n$, and that $\|A\| = \|a\|$, where $\|a\|$ is the Euclidean norm of a.

9. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Suppose that $\|Df(x) - I\| \leq \mu$ for some $\mu \in (0,1)$ and all $x \in \mathbb{R}^n$, where I is the identity map on $\mathbb{R}^n$ and $\|\cdot\|$ is the operator norm. Show that f is an open mapping, i.e. that f maps open subsets to open subsets. Show that $\|x - y\| \leq (1-\mu)^{-1} \|f(x) - f(y)\|$ for all $x, y \in \mathbb{R}^n$, and deduce that f is one-to-one and that $f(\mathbb{R}^n)$ is closed in $\mathbb{R}^n$. Conclude that f is a diffeomorphism of $\mathbb{R}^n$, i.e. that f is a bijection with C^1 inverse. What can you say about a C^1 map $f : \mathbb{R}^n \to \mathbb{R}^n$ assumed to satisfy only that $\|Df(x) - I\| < 1$ for all $x \in \mathbb{R}^n$?

10. Let $C = \{(x,y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$ and define $F: \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x,y) = (x, x^3 + y^3 - 3xy)$. Show that F is locally C^1-invertible around each point of C except $(0,0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$; that is, show that if $(x_0, y_0) \in C \backslash \{(0,0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ then there are open sets U containing (x_0, y_0) and V containing $F(x_0, y_0) = (x_0, 0)$ such that F maps U bijectively to V with inverse a C^1 function. What is the derivative of the inverse function? Deduce that for each point $(x_0, y_0) \in C \setminus \{(0,0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$, there exists an open subset $I \subset \mathbb{R}$ containing x_0 and a C^1 function $g: I \to \mathbb{R}$ such that $C \cap U = \operatorname{graph} g \equiv \{(x, g(x)) : x \in I\}$.

11. Let $\mathcal{M}_n$ be the space of $n \times n$ real matrices equipped with a norm. Show that the determinant function $\det: \mathcal{M}_n \to \mathbb{R}$ is differentiable at the identity matrix I with $D\det(I)(H) = \operatorname{tr}(H)$. Deduce that det is differentiable at any invertible matrix A with $D\det(A)(H) = \det A \operatorname{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $D^2 \det(I)$ as a bilinear map.

12*. (i) Let f be a real-valued C^2 function on an open subset U of $\mathbb{R}^2$. If f has a local maximum at a point $a \in U$ (meaning that there is $\rho > 0$ such that $B_\rho(a) \subset U$ and $f(x) \leq f(a)$ for every $x \in B_\rho(a)$), show that $Df(a) = 0$ and that the matrix $H = (D_{ij} f(a))$ is negative semi-definite (i.e. has non-positive eigenvalues).

(ii) Let U be a bounded open subset of $\mathbb{R}^2$ and let $f\colon \overline{U} \to \mathbb{R}$ be continuous on $\overline{U}$ (the closure of U) and C^2 in U. If f satisfies the partial differential inequality $\Delta f + aD_1 f + bD_2 f + cf \geq 0$ in U where Δ is the Laplace's operator defined by $\Delta f = D_{11} f + D_{22} f$, and a, b, c are real-valued functions on U with $c < 0$ on U, and if f is positive somewhere in $\overline{U}$, show that

$$\sup_{\overline{U}} f = \sup_{\partial U} f$$

where $\partial U = \overline{U} \backslash U$ is the boundary of U. Deduce that if a, b, c are as above, $\varphi\colon \partial U \to \mathbb{R}$ is a given continuous function, then for any $g\colon \mathbb{R}^2 \to \mathbb{R}$ there is at most one continuous function f on $\overline{U}$ that is C^2 in U and solves the boundary value problem $\Delta f + aD_1 f + bD_2 f + cf = g$ in U, $f = \varphi$ on ∂U.