

Part IB — Analysis II

Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Uniform convergence

The general principle of uniform convergence. A uniform limit of continuous functions is continuous. Uniform convergence and termwise integration and differentiation of series of real-valued functions. Local uniform convergence of power series. [3]

Uniform continuity and integration

Continuous functions on closed bounded intervals are uniformly continuous. Review of basic facts on Riemann integration (from Analysis I). Informal discussion of integration of complex-valued and \mathbb{R}^n -valued functions of one variable; proof that $\|\int_a^b f(x) dx\| \leq \int_a^b \|f(x)\| dx$. [2]

\mathbb{R}^n as a normed space

Definition of a normed space. Examples, including the Euclidean norm on \mathbb{R}^n and the uniform norm on $C[a, b]$. Lipschitz mappings and Lipschitz equivalence of norms. The Bolzano-Weierstrass theorem in \mathbb{R}^n . Completeness. Open and closed sets. Continuity for functions between normed spaces. A continuous function on a closed bounded set in \mathbb{R}^n is uniformly continuous and has closed bounded image. All norms on a finite-dimensional space are Lipschitz equivalent. [5]

Differentiation from \mathbb{R}^m to \mathbb{R}^n

Definition of derivative as a linear map; elementary properties, the chain rule. Partial derivatives; continuous partial derivatives imply differentiability. Higher-order derivatives; symmetry of mixed partial derivatives (assumed continuous). Taylor's theorem. The mean value inequality. Path-connectedness for subsets of \mathbb{R}^n ; a function having zero derivative on a path-connected open subset is constant. [6]

Metric spaces

Definition and examples. *Metrics used in Geometry*. Limits, continuity, balls, neighbourhoods, open and closed sets. [4]

The Contraction Mapping Theorem

The contraction mapping theorem. Applications including the inverse function theorem (proof of continuity of inverse function, statement of differentiability). Picard's solution of differential equations. [4]

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0 Introduction

1 Uniform convergence

Definition (Pointwise convergence). The sequence f_n converges *pointwise* to f if

$$f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

for all x .

Definition (Uniform convergence). A sequence of functions $f_n : E \rightarrow \mathbb{R}$ converges *uniformly* to f if

$$(\forall \varepsilon)(\exists N)(\forall x)(\forall n > N) |f_n(x) - f(x)| < \varepsilon.$$

Alternatively, we can say

$$(\forall \varepsilon)(\exists N)(\forall n > N) \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon.$$

Definition (Uniformly Cauchy sequence). A sequence $f_n : E \rightarrow \mathbb{R}$ of functions is *uniformly Cauchy* if

$$(\forall \varepsilon > 0)(\exists N)(\forall m, n > N) \sup_{x \in E} |f_n(x) - f_m(x)| < \varepsilon.$$

2 Series of functions

2.1 Convergence of series

Definition (Convergence of series). Let $g_n; E \rightarrow \mathbb{R}$ be a sequence of functions. Then we say the series $\sum_{n=1}^{\infty} g_n$ converges at a point $x \in E$ if the sequence of partial sums

$$f_n = \sum_{j=1}^n g_j$$

converges at x . The series converges uniformly if f_n converges uniformly.

Definition (Absolute convergence). $\sum g_n$ converges *absolutely* at a point $x \in E$ if $\sum |g_n|$ converges at x .
 $\sum g_n$ converges *absolutely uniformly* if $\sum |g_n|$ converges uniformly.

2.2 Power series

3 Uniform continuity and integration

3.1 Uniform continuity

Definition (Uniform continuity). Let $E \subseteq \mathbb{R}$ and $f : E \rightarrow \mathbb{R}$. We say that f is *uniformly continuous* on E if

$$(\forall \varepsilon)(\exists \delta > 0)(\forall x)(\forall y) |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

3.2 Applications to Riemann integrability

Definition (Riemann integrability). A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is *Riemann integrable* on $[a, b]$ if $I^*(f) = I_*(f)$. We write

$$\int_a^b f(x) \, dx = I^*(f) = I_*(f).$$

Definition (Riemann integrability of vector-valued function). Let $\mathbf{f} : [a, b] \rightarrow \mathbb{R}^n$ be a vector-valued function. Write

$$\mathbf{f}(x) = (f_1(x), f_2(x), \dots, f_n(x))$$

for all $x \in [a, b]$. Then \mathbf{f} is *Riemann integrable* iff $f_j : [a, b] \rightarrow \mathbb{R}$ is integrable for all j . The integral is defined as

$$\int_a^b \mathbf{f}(x) \, dx = \left(\int_a^b f_1(x) \, dx, \dots, \int_a^b f_n(x) \, dx \right) \in \mathbb{R}^n.$$

3.3 Non-examinable fun*

Definition (Lebesgue measure zero*). A subset $A \subseteq \mathbb{R}$ is said to have (*Lebesgue measure zero*) if for any $\varepsilon > 0$, there exists a countable (possibly finite) collection of open intervals I_j such that

$$A \subseteq \bigcup_{j=1}^{\infty} I_j,$$

and

$$\sum_{j=1}^{\infty} |I_j| < \varepsilon.$$

here $|I_j|$ is defined as the length of the interval, not the cardinality (obviously).

4 \mathbb{R}^n as a normed space

4.1 Normed spaces

Definition (Normed space). Let V be a real vector space. A *norm* on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}$ satisfying

- (i) $\|\mathbf{x}\| \geq 0$ with equality iff $\mathbf{x} = \mathbf{0}$ (non-negativity)
- (ii) $\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ (linearity in scalar multiplication)
- (iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

A *normed space* is a pair $(V, \|\cdot\|)$. If the norm is understood, we just say V is a normed space. We do have to be slightly careful since there can be multiple norms on a vector space.

Definition (Lipschitz equivalence of norms). Let V be a (real) vector space. Two norms $\|\cdot\|, \|\cdot\|'$ on V are *Lipschitz equivalent* if there are real constants $0 < a < b$ such that

$$a\|\mathbf{x}\| \leq \|\mathbf{x}\|' \leq b\|\mathbf{x}\|$$

for all $\mathbf{x} \in V$.

It is easy to show this is indeed an equivalence relation on the set of all norms on V .

Definition (Open ball). Let $(V, \|\cdot\|)$ be a normed space, $\mathbf{a} \in V$, $r > 0$. The *open ball* centered at \mathbf{a} with radius r is

$$B_r(\mathbf{a}) = \{\mathbf{x} \in V : \|\mathbf{x} - \mathbf{a}\| < r\}.$$

Definition (Bounded subset). Let $(V, \|\cdot\|)$ be a normed space. A subset $E \subseteq V$ is *bounded* if there is some $R > 0$ such that

$$E \subseteq B_R(\mathbf{0}).$$

Definition (Convergence of sequence). Let $(V, \|\cdot\|)$ be a normed space. A sequence (x_k) in V *converges to* $\mathbf{x} \in V$ if $\|\mathbf{x}_k - \mathbf{x}\| \rightarrow 0$ (as a sequence in \mathbb{R}), i.e.

$$(\forall \varepsilon > 0)(\exists N)(\forall k \geq N) \|\mathbf{x}_k - \mathbf{x}\| < \varepsilon.$$

4.2 Cauchy sequences and completeness

Definition (Cauchy sequence). Let $(V, \|\cdot\|)$ be a normed space. A sequence $(\mathbf{x}^{(k)})$ in V is a *Cauchy sequence* if

$$(\forall \varepsilon)(\exists N)(\forall n, m \geq N) \|\mathbf{x}^{(n)} - \mathbf{x}^{(m)}\| < \varepsilon.$$

Definition (Complete normed space). A normed space $(V, \|\cdot\|)$ is *complete* if every Cauchy sequence converges to an element in V .

Definition (Open set). Let $(V, \|\cdot\|)$ be a normed space. A subspace $E \subseteq V$ is *open* in V if for any $\mathbf{y} \in E$, there is some $r > 0$ such that

$$B_r(\mathbf{y}) = \{\mathbf{x} \in V : \|\mathbf{x} - \mathbf{y}\| < r\} \subseteq E.$$

Definition (Limit point). Let $(V, \|\cdot\|)$ be a normed space, $E \subseteq V$. A point $\mathbf{y} \in V$ is a *limit point* of E if there is a sequence (\mathbf{x}_k) in E with $\mathbf{x}_k \neq \mathbf{y}$ for all k and $\mathbf{x}_k \rightarrow \mathbf{y}$.

Definition (Closed set). Let $(V, \|\cdot\|)$ be a normed space. Then $E \subseteq V$ is *closed* if $V \setminus E$ is open, i.e. E contains all its limit points.

4.3 Sequential compactness

Definition ((Sequentially) compact set). Let V be a normed vector space. A subset $K \subseteq V$ is said to be *compact* (or *sequentially compact*) if every sequence in K has a subsequence that converges to a point in K .

4.4 Mappings between normed spaces

Definition (Continuity of mapping). Let $\mathbf{y} \in E$. We say $f : E \rightarrow V'$ is *continuous at \mathbf{y}* if for all $\varepsilon > 0$, there is $\delta > 0$ such that the following holds:

$$(\forall \mathbf{x} \in E) \|\mathbf{x} - \mathbf{y}\|_V < \delta \Rightarrow \|f(\mathbf{x}) - f(\mathbf{y})\|_{V'} < \varepsilon.$$

Definition (Continuous function). $f : E \rightarrow V'$ is *continuous* if f is continuous at every point $\mathbf{y} \in E$.

5 Metric spaces

5.1 Preliminary definitions

Definition (Metric space). Let X be any set. A *metric* on X is a function $d : X \times X \rightarrow \mathbb{R}$ that satisfies

- $d(x, y) \geq 0$ with equality iff $x = y$ (non-negativity)
- $d(x, y) = d(y, x)$ (symmetry)
- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

The pair (X, d) is called a *metric space*.

Definition (Lipschitz equivalent metrics). Metrics d, d' on a set X are said to be *Lipschitz equivalent* if there are (positive) constants A, B such that

$$Ad(x, y) \leq d'(x, y) \leq Bd(x, y)$$

for all $x, y \in X$.

Definition (Metric subspace). Given a metric space (X, d) and a subset $Y \subseteq X$, the restriction $d|_{Y \times Y} \rightarrow \mathbb{R}$ is a metric on Y . This is called the *induced metric* or *subspace metric*.

Definition (Convergence). Let (X, d) be a metric space. A sequence $x_n \in X$ is said to *converge* to x if $d(x_n, x) \rightarrow 0$ as a real sequence. In other words,

$$(\forall \varepsilon)(\exists K)(\forall k > K) d(x_k, x) < \varepsilon.$$

Alternatively, this says that given any ε , for sufficiently large k , we get $x_k \in B_\varepsilon(x)$.

5.2 Topology of metric spaces

Definition (Open subset). Let (X, d) be a metric space. A subset $U \subseteq X$ is *open* if for every $y \in U$, there is some $r > 0$ such that $B_r(y) \subseteq U$.

Definition (Topology). Let (X, d) be a metric space. The *topology* on (X, d) is the collection of open subsets of X . We say it is the topology induced by the metric.

Definition (Topological notion). A notion or property is said to be a *topological* notion or property if it only depends on the topology, and not the metric.

Definition (Neighbourhood). Given a metric space X and a point $x \in X$, a *neighbourhood* of x is an open set containing x .

Definition (Limit point). Let (X, d) be a metric space and $E \subseteq X$. A point $y \in X$ is a *limit point* of E if there exists a sequence $x_k \in E$, $x_k \neq y$ such that $x_k \rightarrow y$.

Definition (Closed subset). A subset $E \subseteq X$ is *closed* if E contains all its limit points.

5.3 Cauchy sequences and completeness

Definition (Cauchy sequence). Let (X, d) be a metric space. A sequence (x_n) in X is *Cauchy* if

$$(\forall \varepsilon)(\exists N)(\forall n, m, m \geq N) d(x_n, x_m) < \varepsilon.$$

Definition (Complete metric space). A metric space is complete if all Cauchy sequences converge.

5.4 Compactness

Definition ((Sequential) compactness). A metric space (X, d) is (*sequentially*) *compact* if every sequence in X has a convergent subsequence.

A subset $K \subseteq X$ is said to be compact if $(K, d|_{K \times K})$ is compact. In other words, K is compact if every sequence in K has a subsequence that converges to some point in K .

Definition (Totally bounded*). A metric space (X, d) is said to be *totally bounded* if for all $\varepsilon > 0$, there is an integer $N \in \mathbb{N}$ and points $x_1, \dots, x_N \in X$ such that

$$X = \bigcup_{i=1}^N B_\varepsilon(x_i).$$

5.5 Continuous functions

Definition (Continuity). Let (X, d) and (X', d') be metric spaces. A function $f : X \rightarrow X'$ is *continuous at* $y \in X$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x) d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon.$$

This is true if and only if for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$B_\delta(y) \subseteq f^{-1}B_\varepsilon(f(y)).$$

f is *continuous* if f is continuous at each $y \in X$.

Definition (Uniform continuity). f is *uniformly continuous* on X if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in X) d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon.$$

This is true if and only if for all ε , there is some δ such that for *all* y , we have

$$B_\delta(y) \subseteq f^{-1}(B_\varepsilon(f(y))).$$

Definition (Lipschitz function and Lipschitz constant). f is said to be *Lipschitz* on X if there is some $K \in [0, \infty)$ such that for all $x, y \in X$,

$$d'(f(x), f(y)) \leq Kd(x, y)$$

Any such K is called a Lipschitz constant.

5.6 The contraction mapping theorem

Definition (Contraction mapping). Let (X, d) be metric space. A mapping $f : X \rightarrow X$ is a *contraction* if there exists some λ with $0 \leq \lambda < 1$ such that

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

Notation. For $\mathbf{x}_0 \in \mathbb{R}^n$, $R > 0$, we let

$$\overline{B_R(\mathbf{x}_0)} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\|_2 \leq R\}.$$

6 Differentiation from \mathbb{R}^m to \mathbb{R}^n

6.1 Differentiation from \mathbb{R}^m to \mathbb{R}^n

Definition (Limit of function). Let $E \subseteq \mathbb{R}^n$ and $f : E \rightarrow \mathbb{R}^m$. Let $\mathbf{a} \in \mathbb{R}^n$ be a limit point of E , and let $\mathbf{b} \in \mathbb{R}^m$. We say

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = \mathbf{b}$$

if for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$(\forall \mathbf{x} \in E) 0 < \|\mathbf{x} - \mathbf{a}\| < \delta \Rightarrow \|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon.$$

Definition (Differentiation in \mathbb{R}^n). Let $U \subseteq \mathbb{R}^n$ be open, $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$. We say \mathbf{f} is differentiable at a point $\mathbf{a} \in U$ if there exists a linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}.$$

We call A the *derivative of \mathbf{f} at \mathbf{a}* . We write the derivative as $D\mathbf{f}(\mathbf{a})$.

Notation. We write $L(\mathbb{R}^n; \mathbb{R}^m)$ for the space of linear maps $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Notation (Little o notation). For any function $\alpha : B_r(0) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, write

$$\alpha(\mathbf{h}) = o(\mathbf{h})$$

if

$$\frac{\alpha(\mathbf{h})}{\|\mathbf{h}\|} \rightarrow \mathbf{0} \text{ as } \mathbf{h} \rightarrow \mathbf{0}.$$

In other words, $\alpha \rightarrow \mathbf{0}$ faster than $\|\mathbf{h}\|$ as $\mathbf{h} \rightarrow \mathbf{0}$.

Note that officially, $\alpha(\mathbf{h}) = o(\mathbf{h})$ as a whole is a piece of notation, and does not represent equality.

Definition (Directional derivative). We write

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{a}) = \lim_{t \rightarrow 0} \frac{\mathbf{f}(\mathbf{a} + t\mathbf{u}) - \mathbf{f}(\mathbf{a})}{t}$$

whenever this limit exists. We call $D_{\mathbf{u}}\mathbf{f}(\mathbf{a})$ the *directional derivative of \mathbf{f} at $\mathbf{a} \in U$ in the direction of $\mathbf{u} \in \mathbb{R}^n$* .

By definition, we have

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{a}) = \left. \frac{d}{dt} \right|_{t=0} \mathbf{f}(\mathbf{a} + t\mathbf{u}).$$

Definition (Partial derivative). The j th *partial derivative* of $f : U \rightarrow \mathbb{R}$ at $\mathbf{a} \in U$ is

$$D_{\mathbf{e}_j}f(\mathbf{a}) = \lim_{t \rightarrow 0} \frac{f(\mathbf{a} + t\mathbf{e}_j) - f(\mathbf{a})}{t},$$

when the limit exists. We often write this as

$$D_{\mathbf{e}_j}f(\mathbf{a}) = D_jf(\mathbf{a}) = \frac{\partial f}{\partial x_j}.$$

6.2 The operator norm

Definition (Operator norm). The *operator norm* on $\mathcal{L} = L(\mathbb{R}^n; \mathbb{R}^m)$ is defined by

$$\|A\| = \sup_{\mathbf{x} \in \mathbb{R}^n: \|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

6.3 Mean value inequalities

Definition (Path-connected subset). A subset $E \subseteq \mathbb{R}^n$ is *path-connected* if for any $\mathbf{a}, \mathbf{b} \in E$, there is a continuous map $\gamma : [0, 1] \rightarrow E$ such that

$$\gamma(0) = \mathbf{a}, \quad \gamma(1) = \mathbf{b}.$$

6.4 Inverse function theorem

Definition (C^1 function). Let $U \subseteq \mathbb{R}^n$ be open. We say $\mathbf{f} : U \rightarrow \mathbb{R}^m$ is C^1 on U if \mathbf{f} is differentiable at each $\mathbf{x} \in U$ and

$$D\mathbf{f} : U \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$$

is continuous.

We write $C^1(U)$ or $C^1(U; \mathbb{R}^m)$ for the set of all C^1 maps from U to \mathbb{R}^m .

Definition (Diffeomorphism). Let $U, U' \subseteq \mathbb{R}^n$ be open, then a map $\mathbf{g} : U \rightarrow U'$ is a *diffeomorphism* if it is C^1 with a C^1 inverse.

6.5 2nd order derivatives

Definition (2nd derivative). Let $U \subseteq \mathbb{R}^n$ be open, $\mathbf{f} : U \rightarrow \mathbb{R}^m$ be differentiable. Then $D\mathbf{f} : U \rightarrow L(\mathbb{R}^n; \mathbb{R}^m)$. We say $D\mathbf{f}$ is *differentiable* at $\mathbf{a} \in U$ if there exists $A \in L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m))$ such that

$$\lim_{\mathbf{h} \rightarrow 0} \frac{1}{\|\mathbf{h}\|} (D\mathbf{f}(\mathbf{a} + \mathbf{h}) - D\mathbf{f}(\mathbf{a}) - A\mathbf{h}) = 0.$$

Notation. Write

$$D_{ij}\mathbf{f}(\mathbf{a}) = D_i(D_j\mathbf{f})(\mathbf{a}) = \frac{\partial^2}{\partial x_i \partial x_j} \mathbf{f}(\mathbf{a}).$$

Notation. We define $D^2\mathbf{f}(\mathbf{a}) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$D^2\mathbf{f}(\mathbf{a})(\mathbf{u}, \mathbf{v}) = D(D\mathbf{f})(\mathbf{a})(\mathbf{u})(\mathbf{v}).$$