B13a **Numerical Analysis: Example Sheet 1** Lent 2016

*A $*$ denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are **not** necessarily harder than unstarred questions.*

Corrections and suggestions should be emailed to `G.Moore@maths.cam.ac.uk`.

1. Suppose that the function values $f(0)$, $f(1)$, $f(2)$ and $f(3)$ are given and that we wish to estimate
$$f(6), \quad f'(0) \quad \text{and} \quad \int_0^3 f(x)\,\mathrm{d}x.$$
One method is to let p be the cubic polynomial that interpolates these function values, and then to employ the approximants
$$p(6), \quad p'(0) \quad \text{and} \quad \int_0^3 p(x)\,\mathrm{d}x$$
respectively. Deduce from the Lagrange formula for p that each approximant is a linear combination of the four data with constant coefficients. Calculate the numerical values of these constants. Verify your work by showing that the approximants are exact when f is an arbitrary cubic polynomial.

2. Let f be a function in $C^4[0,1]$ and let p be a cubic polynomial that interpolates $f(0)$, $f'(0)$, $f(1)$ and $f'(1)$. Deduce from the Rolle theorem that for every $x \in [0,1]$ there exists $\xi \in [0,1]$ such that the equation
$$f(x) - p(x) = \tfrac{1}{24}x^2(x-1)^2 f^{(4)}(\xi)$$
is satisfied.

3. Let a, b and c be distinct real numbers (not necessarily in ascending order), and let $f(a)$, $f(b)$, $f'(a)$, $f'(b)$ and $f'(c)$ be given. Because there are five data, one might try to approximate f by a polynomial of degree at most four that interpolates the data. Prove by a general argument that this interpolation problem has a solution and that the solution is unique, if and only if there is no nonzero polynomial $p \in \mathbb{P}_4[x]$ that satisfies $p(a) = p(b) = p'(a) = p'(b) = p'(c) = 0$. Hence, given a and b, show that there exists a unique value of $c \neq a, b$ such that there is no unique solution.

   Note: This form of interpolation when both function values and derivatives are fitted, perhaps at different points, is known as Birkhoff–Hermite interpolation.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function and let p be the polynomial of degree at most n that interpolates f at the pairwise distinct points $x_0, x_1, \ldots, x_n$. Further, let x be any real number that is not an interpolation point. Deduce the identity
$$f(x) - p(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{j=0}^{n} (x - x_j)$$
from the definition of the divided difference $f[x_0, x_1, \ldots, x_n, x]$.

5. Simulating a computer that works to only four decimal places, form the table of divided differences of the values $f(0) = 0$, $f(0.1) = 0.0998$, $f(0.4) = 0.3894$ and $f(0.7) = 0.6442$ of $\sin x$. Hence identify the polynomial that is given by Newton's interpolation method. Due to rounding errors, this polynomial should differ from the one that would be given by exact arithmetic. Take the view, however, that the *computed* values of $f[0.0, 0.1]$, $f[0.0, 0.1, 0.4]$ and $f[0.0, 0.1, 0.4, 0.7]$ and the function value $f(0)$ are correct. Then, by working backwards through the difference table, identify the values of $f(0)$, $f(0.1)$, $f(0.4)$ and $f(0.7)$ that would give these divided differences in exact arithmetic.

6. Set $f(x) = 2x - 1$, $x \in [0, 1]$. We require a function of form

$$p(x) = \sum_{k=0}^{n} a_k \cos(k\pi x), \quad 0 \leqslant x \leqslant 1,$$

that satisfies the condition

$$\int_0^1 [f(x) - p(x)]^2 \, \mathrm{d}x < 10^{-4}.$$

Explain why it is sufficient if the value of $a_0^2 + \frac{1}{2}\sum_{k=1}^{n} a_k^2$ exceeds $\frac{1}{3} - 10^{-4}$, where the coefficients $\{a_k\}_{k=0}^{n}$ are calculated to minimize this integral. Hence find the smallest acceptable value of n.

7. The polynomials $\{p_n\}_{n \in \mathbb{Z}^+}$ are defined by the three-term recurrence formula

$$\begin{aligned} p_0(x) &\equiv 1, \\ p_1(x) &= 2x, \\ p_{n+1}(x) &= 2x p_n(x) - p_{n-1}(x), \qquad n = 1, 2, \ldots. \end{aligned}$$

Prove that they are orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \sqrt{1 - x^2} \, \mathrm{d}x$$

and evaluate $\langle p_n, p_n \rangle$ for $n \in \mathbb{Z}^+$. *Hint: Prove that* $p_n(x) = \sin(n+1)\theta / \sin\theta$, *where* $x = \cos\theta$.

Note: These p_n*s are known as* Chebyshev polynomials of the second kind *and denoted by* $p_n = U_n$.

8. Calculate the coefficients b_1, b_2, c_1 and c_2 so that the approximant

$$\int_0^1 f(x) \, \mathrm{d}x \approx b_1 f(c_1) + b_2 f(c_2)$$

is exact when f is a cubic polynomial. You may exploit the fact that c_1 and c_2 are the zeros of a quadratic polynomial that is orthogonal to all linear polynomials. Verify your calculation by testing the formula when $f(x) = 1, x, x^2$ and x^3.

9. The functions $p_0, p_1, p_2, \ldots$ are generated by the Rodrigues formula

$$p_n(x) = \mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n \mathrm{e}^{-x}), \qquad 0 \leqslant x < \infty.$$

Show that these functions are polynomials and prove by integration by parts that for every $p \in \mathbb{P}_{n-1}[x]$ we have the orthogonality condition $\langle p_n, p \rangle = 0$ with respect to the scalar product

$$\langle f, g \rangle := \int_0^\infty \mathrm{e}^{-x} f(x) g(x) \, \mathrm{d}x.$$

Derive the coefficients of p_3, p_4 and p_5 from the Rodrigues formula. Verify that these coefficients are compatible with a three term recurrence relation of the form

$$p_5(x) = (\gamma x - \alpha) p_4(x) - \beta p_3(x), \qquad x \in \mathbb{R},$$

where α, β and γ are constants.

Note: $L_n = \frac{1}{n!} p_n$ *or, if you want to be really sophisticated,* $L_n^{(0)} = \frac{1}{n!} p_n$, *are known as* Laguerre polynomials.

10. Let $p(\frac{1}{2}) = \frac{1}{2}(f(0) + f(1))$, where f is a function in $C^2[0,1]$. Find the least constants c_0, c_1 and c_2 such that the error bounds

$$|f(\tfrac{1}{2}) - p(\tfrac{1}{2})| \leqslant c_k \|f^{(k)}\|_\infty \,, \qquad k = 0, 1, 2,$$

are valid.

Note: The cases $k = 0$ and $k = 1$ are easy if one works from first principles, and the Peano kernel theorem is suitable when $k = 2$. Also try the Peano kernel theorem when $k = 1$.

∗11. Express the divided difference $f[0,1,2,4]$ in the form

$$f[0,1,2,4] = \tfrac{1}{2}\int_0^4 K(\theta) f'''(\theta)\, \mathrm{d}\theta,$$

assuming that f''' exists and is continuous. Sketch the kernel function $K(\theta)$ for $0 \leqslant \theta \leqslant 4$. By integrating $K(\theta)$ analytically and using the mean value theorem prove that

$$f[0,1,2,4] = \tfrac{1}{6} f'''(\xi)$$

for some point $\xi \in [0,4]$.

Note: Another proof of this result was given in the lecture on divided differences.

12. Let f be a function in $C^4[0,1]$ and let ξ be any fixed point in $[0,1]$. Calculate the coefficients α, β, γ and δ such that the approximant

$$f'''(\xi) \approx \alpha f(0) + \beta f(1) + \gamma f'(0) + \delta f'(1)$$

is exact for all cubic polynomials. Prove that the inequality

$$\left| f'''(\xi) - \alpha f(0) - \beta f(1) - \gamma f'(0) - \delta f'(1) \right| \leqslant \left\{ \tfrac{1}{2} - \xi + 2\xi^3 - \xi^4 \right\} \|f^{(4)}\|_\infty$$

is satisfied. Show that this inequality holds as an equation if we allow f to be the function

$$f(x) = \begin{cases} -(x-\xi)^4, & 0 \leqslant x \leqslant \xi, \\ (x-\xi)^4, & \xi \leqslant x \leqslant 1. \end{cases}$$

∗13. Given f and g in $C[a,b]$, let $h := fg$. Prove by induction that the divided differences of h satisfy the equation

$$h[x_0, x_1, \ldots, x_n] = \sum_{j=0}^{n} f[x_0, x_1, \ldots, x_j]\, g[x_j, x_{j+1}, \ldots, x_n].$$

By expressing the differences in terms of derivatives and by letting the points $x_0, x_1, \ldots, x_n$ become coincident, deduce the Leibniz formula for the nth derivative of a product of two functions.

B13b Numerical Analysis: Example Sheet 2 Lent 2016

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1. Let $h = 1/M$, where $M \geqslant 1$ is an integer. Consider the differential equations

$$y' = -\frac{y}{1+t} \quad \text{and} \quad y' = \frac{2y}{1+t}, \qquad 0 \leqslant t \leqslant 1, \tag{$*$}$$

with initial condition $y(0) = 1$ in both cases. In each case find the exact solution of the differential equation.

Define Euler's method, and apply it to calculate the estimates $\{\boldsymbol{y}_n\}_{n=1,2,\ldots,M}$ of $\boldsymbol{y}(nh)$ for each of the equations letting $y_0 = y(0) = 1$. By using induction and by cancelling as many terms as possible in the resultant products, deduce simple explicit expressions for y_n, $n = 1, 2, \ldots, M$, which should be free from summations and products of n terms. Deduce the exact solutions of the equations from the limit $h \to 0$. Verify that the magnitude of the errors $y_n - y(nh)$, $n = 1, 2, \ldots, M$, is at most $\mathcal{O}(h)$.

2. Assuming that $\boldsymbol{f}$ satisfies the Lipschitz condition and the true solution possesses a bounded third derivative in $[0, t^*]$, apply the method of analysis of the Euler method, given in the lectures, to prove that the trapezoidal rule

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \tfrac{1}{2}h[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})]$$

converges and that $\|\boldsymbol{y}_n - \boldsymbol{y}(t_n)\| \leqslant ch^2$ for some $c > 0$ and all n such that $0 \leqslant nh \leqslant t^*$.

3. The s-step Adams–Bashforth method is of order s and has the form

$$\boldsymbol{y}_{n+s} = \boldsymbol{y}_{n+s-1} + h \sum_{j=0}^{s-1} \sigma_j \boldsymbol{f}(t_{n+j}, \boldsymbol{y}_{n+j}).$$

Calculate the actual values of the coefficients in the case $s = 3$

4. By solving a three-term recurrence relation, calculate analytically the sequence of values $\{\boldsymbol{y}_n : n = 2, 3, 4, \ldots\}$ that is generated by the *explicit midpoint rule*

$$\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + 2h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}),$$

when it is applied to the ODE $y' = -y$, $t \geqslant 0$. Starting from the values $y_0 = 1$ and $y_1 = 1 - h$, show that the sequence diverges as $n \to \infty$ for *all* $h > 0$.

However if the order of the method is greater or equal to one, the root condition and suitable starting conditions imply convergence in a *finite* interval. Prove that the above implementation of the explicit midpoint rule is consistent with this theorem.

Hint: in the last part, relate the roots of the recurrence relation to $\pm e^{\mp h} + \mathcal{O}(h^3)$.

5. Show that the multistep method

$$\sum_{j=0}^{3} \rho_j \boldsymbol{y}_{n+j} = h \sum_{j=0}^{2} \sigma_j \boldsymbol{f}(t_{n+j}, \boldsymbol{y}_{n+j}), \quad \text{where } \rho_3 = 1 \text{ (as in lectures)},$$

is fourth order only if the conditions $\rho_0 + \rho_2 = 8$ and $\rho_1 = -9$ are satisfied. Hence deduce that this method cannot be both fourth order and satisfy the root condition

$*$6. An s-stage explicit Runge–Kutta method of order s with constant step size $h > 0$ is applied to the differential equation $y' = \lambda y$, $t \geqslant 0$. Prove the identity

$$y_n = \left[\sum_{\ell=0}^{s} \frac{1}{\ell!}(h\lambda)^\ell\right]^n y_0, \qquad n = 0, 1, 2, \ldots,$$

by showing by induction that $k_i = \lambda y_n p_{i-1}(\lambda h)$ where the p_{i-1} are polynomials of degree $i - 1$, deducing that $y_{n+1} = p_s(\lambda h) y_n$, and using the order condition to determine p_s; or otherwise.

7. The following four-stage Runge–Kutta method has order four,

$$\begin{aligned} \boldsymbol{k}_1 &= \boldsymbol{f}(t_n, \boldsymbol{y}_n) \\ \boldsymbol{k}_2 &= \boldsymbol{f}(t_n + \tfrac{1}{3}h, \boldsymbol{y}_n + \tfrac{1}{3}h\boldsymbol{k}_1) \\ \boldsymbol{k}_3 &= \boldsymbol{f}(t_n + \tfrac{2}{3}h, \boldsymbol{y}_n - \tfrac{1}{3}h\boldsymbol{k}_1 + h\boldsymbol{k}_2) \\ \boldsymbol{k}_4 &= \boldsymbol{f}(t_n + h, \boldsymbol{y}_n + h\boldsymbol{k}_1 - h\boldsymbol{k}_2 + h\boldsymbol{k}_3) \\ \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + h(\tfrac{1}{8}\boldsymbol{k}_1 + \tfrac{3}{8}\boldsymbol{k}_2 + \tfrac{3}{8}\boldsymbol{k}_3 + \tfrac{1}{8}\boldsymbol{k}_4). \end{aligned}$$

By considering the equation $y' = y$, show that the order is at most four. Then, *for scalar functions,* prove that the order is at least four in the easy case when f is independent of y, and that the order is at least three in the relatively easy case when f is independent of t.

Comment: do not derive all of the (gory) details when $f(t, y)$ depends on both t and $\boldsymbol{y}$.

8. Find $\mathcal{D} \cap \mathbb{R}$, the intersection of the linear stability domain $\mathcal{D}$ with the real axis, for the following methods:

(1) $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$ (2) $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})]$

(3) $\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + 2h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})$ (4) $\boldsymbol{y}_{n+2} = \boldsymbol{y}_{n+1} + \frac{1}{2}h[3\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) - \boldsymbol{f}(t_n, \boldsymbol{y}_n)]$

(5) The RK method $\boldsymbol{k}_1 = \boldsymbol{f}(t_n, \boldsymbol{y}_n)$, $\boldsymbol{k}_2 = \boldsymbol{f}(t_n + h, \boldsymbol{y}_n + h\boldsymbol{k}_1)$, $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h(\boldsymbol{k}_1 + \boldsymbol{k}_2)$.

Hint: note that to solve $a < X < b$, instead of manipulating the inequalities it can be easier to solve $X = a$ and $X = b$, and then decide which region is required by considering one interior point (as for conformal maps).

9. Show that, if z is a nonzero complex number that is on the boundary of the linear stability domain of the two-step BDF method

$$\boldsymbol{y}_{n+2} - \tfrac{4}{3}\boldsymbol{y}_{n+1} + \tfrac{1}{3}\boldsymbol{y}_n = \tfrac{2}{3}hf(t_{n+2}, \boldsymbol{y}_{n+2})$$

then the real part of z is positive. Thus deduce that this method is A-stable.

Hint: if z is on the boundary of the linear stability domain then $y_n = \exp(in\theta)$ for some real θ.

10. The (stiff) differential equation

$$y'(t) = -10^4(y - t^{-1}) - t^{-2}, \qquad t \geqslant 1, \qquad y(1) = 1,$$

has the analytic solution $y(t) = t^{-1}$, $t \geqslant 1$. Let it be solved numerically by Euler's method $y_{n+1} = y_n + h_n f(t_n, y_n)$ and the backward Euler method $y_{n+1} = y_n + h_n f(t_{n+1}, y_{n+1})$, where $h_n = t_{n+1} - t_n$ is allowed to depend on n and to be different in the two cases. Suppose that, for any $t_n \geqslant 1$, we have $|y_n - y(t_n)| \leqslant 10^{-6}$, and that we require $|y_{n+1} - y(t_{n+1})| \leqslant 10^{-6}$. Show that Euler's method can fail if $h_n = 2 \times 10^{-4}$, but that the backward Euler method always succeeds if $h_n \leqslant 10^{-2} t_n t_{n+1}^2$.

Hint: find relations between $y_{n+1} - y(t_{n+1})$ and $y_n - y(t_n)$ for general y_n and t_n.

11. This question concerns the predictor-corrector pair

$$\begin{aligned} \boldsymbol{y}^{\mathrm{P}}_{n+3} &= -\tfrac{1}{2}\boldsymbol{y}_n + 3\boldsymbol{y}_{n+1} - \tfrac{3}{2}\boldsymbol{y}_{n+2} + 3h\boldsymbol{f}(t_{n+2}, \boldsymbol{y}_{n+2}), \\ \boldsymbol{y}^{\mathrm{C}}_{n+3} &= \tfrac{1}{11}[2\boldsymbol{y}_n - 9\boldsymbol{y}_{n+1} + 18\boldsymbol{y}_{n+2} + 6h\boldsymbol{f}(t_{n+3}, \boldsymbol{y}_{n+3})]. \end{aligned}$$

Show that both methods are third order, and that the estimate of the error of the corrector formula by Milne's device has the value $\frac{6}{17}|\boldsymbol{y}^{\mathrm{P}}_{n+3} - \boldsymbol{y}^{\mathrm{C}}_{n+3}|$.

12. Let $u(x)$, $0 \leqslant x \leqslant 1$, be a six-times differentiable function that satisfies the ODE $u''(x) = f(x)$, $0 \leqslant x \leqslant 1$, $u(0)$ and $u(1)$ being given. Further, we let $x_m = mh = m/M$, $m = 0, 1, \ldots, M$, for some positive integer M, and calculate the estimates $u_m \approx u(x_m)$, $m = 1, 2, \ldots, M-1$, by solving the difference equation

$$u_{m-1} - 2u_m + u_{m+1} = h^2 f(x_m) + \alpha h^2[f(x_{m-1}) - 2f(x_m) + f(x_{m+1})], \qquad m = 1, 2, \ldots, M-1,$$

where $u_0 = u(0)$, $u_M = u(1)$, and α is a positive parameter. Show that there exists a choice of α such that the local truncation error of the difference equation is $\mathcal{O}(h^6)$.

B13c **Numerical Analysis: Example Sheet 3** Lent 2016

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1. Calculate *all* LU factorizations of the matrix

$$A = \begin{bmatrix} 10 & 6 & -2 & 1 \\ 10 & 10 & -5 & 0 \\ -2 & 2 & -2 & 1 \\ 1 & 3 & -2 & 3 \end{bmatrix},$$

where all diagonal elements of L are one. By using one of these factorizations, find *all* solutions of the equation $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b}^\top = [-2, 0, 2, 1]$.

2. By using column pivoting if necessary to exchange rows of A, an LU factorization of a real $n \times n$ matrix A is calculated, where L has ones on its diagonal, and where the moduli of the off-diagonal elements of L do not exceed one. Let α be the largest of the moduli of the elements of A. Prove by induction on i that elements of U satisfy the condition $|u_{ij}| \leqslant 2^{i-1}\alpha$. Then construct 2×2 and 3×3 nonzero matrices A that yield $|u_{22}| = 2\alpha$ and $|u_{33}| = 4\alpha$ respectively.

3. Let A be a real $n \times n$ matrix that has the factorization $A = LU$, where L is lower triangular with ones on its diagonal and U is upper triangular. Prove that, for every integer $k \in \{1, 2, \ldots, n\}$, the first k rows of U span the same space as the first k rows of A. Prove also that the first k columns of A are in the k-dimensional subspace that is spanned by the first k columns of L.

   Hence deduce that no LU factorization of the given form exists if we have $\operatorname{rank} H_k < \operatorname{rank} B_k$, where H_k is the leading $k \times k$ submatrix of A and where B_k is the $n \times k$ matrix whose columns are the first k columns of A.

   Further, deduce that if A is non-singular and the LU factorisation exists, then all leading zeros to the left of the diagonal in the rows of A are inherited by L, and all leading zeros above the diagonal in the columns of A are inherited by U.

4. Calculate the Cholesky factorization of the matrix

$$\begin{bmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 5 & 1 \\ & & & & 1 & \lambda \end{bmatrix}.$$

   Deduce from the factorization the value of λ that makes the matrix singular. Also find this value of λ by seeking the vector in the null-space of the matrix whose first component is one.

5. Let A be an $n \times n$ nonsingular band matrix that satisfies the condition $a_{ij} = 0$ if $|i-j| > r$, where r is small, and let Gaussian elimination *with column pivoting* be used to solve $Ax = b$. Identify all the coefficients of the intermediate equations that can become nonzero. Hence deduce that the total number of additions and multiplications of the complete calculation can be bounded by a constant multiple of nr^2.

6. Let a_1, a_2 and a_3 denote the columns of the matrix

$$A = \begin{bmatrix} 6 & 6 & 1 \\ 3 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

   Apply the Gram–Schmidt procedure to A, which generates orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2$ and $\boldsymbol{q}_3$. Note that this calculation provides real numbers r_{jk} such that $\boldsymbol{a}_k = \sum_{j=1}^{k} r_{jk}\boldsymbol{q}_j$, $k = 1,2,3$. Hence express A as the product $A = QR$, where Q and R are orthogonal and upper-triangular matrices respectively.

7. Calculate the QR factorization of the matrix of question 6 by using three Givens rotations. Explain why the initial rotation can be any one of the three types $\Omega^{(1,2)}$, $\Omega^{(1,3)}$ and $\Omega^{(2,3)}$. Prove that the final factorization is independent of this initial choice in exact arithmetic, provided that we satisfy the condition that in each row of R the leading nonzero element is positive.

8. Let A be an $n \times n$ matrix, and for $i = 1, 2, \ldots, n$ let $k(i)$ be the number of zero elements in the i-th row of A that come before all nonzero elements in this row and before the diagonal element a_{ii}. Show that the QR factorization of A can be calculated by using at most $\frac{1}{2}n(n-1) - \sum k(i)$ Givens rotations. Hence show that, if A is an upper triangular matrix except that there are nonzero elements in its first column, i.e. $a_{ij} = 0$ when $2 \leqslant j < i \leqslant n$, then its QR factorization can be calculated by using only $2n-3$ Givens rotations.

   Hint: you should find the order of the first $(n-2)$ rotations that brings your matrix to the form considered above.

9. Calculate the QR factorization of the matrix of question 6 by using two Householder reflections. Show that, if this technique is used to generate the QR factorization of a general $n \times n$ matrix A, then the computation can be organised so that the total number of additions and multiplications is bounded above by a constant multiple of n^3.

10. Let

$$A = \begin{bmatrix} 3 & 4 & 7 & -2 \\ 5 & 4 & 9 & 3 \\ 1 & -1 & 0 & 3 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 11 \\ 29 \\ 16 \\ 10 \end{bmatrix}.$$

   Calculate the QR factorization of A by using Householder reflections. In this case A is singular and you should choose Q so that the last row of R is zero. Hence identify all the least squares solutions of the inconsistent system $A\boldsymbol{x} = \boldsymbol{b}$, where we require $\boldsymbol{x}$ to minimize $\|A\boldsymbol{x} - \boldsymbol{b}\|_2$. Verify that all the solutions give the same vector of residuals $A\boldsymbol{x} - \boldsymbol{b}$, and that this vector is orthogonal to the columns of A. There is no need to calculate the elements of Q explicitly.