Part IB GEOMETRY (Lent 2016): Example Sheet 1

(a.g.kovalev@dpmms.cam.ac.uk)

1. Suppose that H is a hyperplane in Euclidean n-space $\mathbb{R}^n$ defined by $\mathbf{u} \cdot \mathbf{x} = c$ for some unit vector $\mathbf{u}$ and constant c. The reflection in H is the map from $\mathbb{R}^n$ to itself given by $\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{u} \cdot \mathbf{x} - c)\mathbf{u}$. Show that this is an isometry. Letting P, Q be points of $\mathbb{R}^n$, show that there is a reflection in some hyperplane that maps P to Q.

2. Suppose that l_1 and l_2 are non-parallel lines in the Euclidean plane $\mathbb{R}^2$, and that r_i denotes the reflection of $\mathbb{R}^2$ in the line l_i, for $i = 1, 2$. Show that the composite $r_1 r_2$ is a rotation of $\mathbb{R}^2$, and describe (in terms of the lines l_1 and l_2) the resulting fixed point and the angle of rotation.

3. Let $R(P, \theta)$ denote the clockwise rotation of $\mathbb{R}^2$ through an angle θ about a point P. If A, B, C are the vertices, labelled clockwise, of a triangle in $\mathbb{R}^2$, prove that $R(A, \theta)R(B, \phi)R(C, \psi)$ is the identity if and only if $\theta = 2\alpha$, $\phi = 2\beta$ and $\psi = 2\gamma$, where α, β, γ denote the angles at, respectively, the vertices A, B, C of the triangle ABC.

4. Show from first principles that a (continuous) curve of shortest length between two points in Euclidean space is a straight line segment, parametrized monotonically.

5. Prove that any isometry of the unit sphere is induced from an isometry of $\mathbb{R}^3$ which fixes the origin. Prove that any matrix $A \in O(3, \mathbb{R})$ is the product of at most three reflections in planes through the origin. Deduce that an isometry of the unit sphere can be expressed as the product of at most three reflections in spherical lines. What isometries are obtained from the product of two reflections? What isometries are obtained from the product of three reflections?

6. By repeatedly applying the result from Question 1, when P is either $\mathbf{0}$ or one of the standard basis vectors of $\mathbb{R}^n$, deduce that any isometry T of $\mathbb{R}^n$ can be written as a composition of at most $n + 1$ reflections.

7. Suppose that P is a point on the unit sphere S^2. For fixed ρ, with $0 < \rho < \pi$, the spherical circle with centre P and radius ρ is the set of points $Q \in S^2$ whose spherical distance from P is ρ. Prove that a spherical circle of radius ρ on S^2 has circumference $2\pi \sin \rho$ and area $2\pi(1 - \cos \rho)$.

8. Given a spherical line l on the sphere S^2 and a point P not on l, show that there is a spherical line l' passing through P and intersecting l at right-angles. Prove that the minimum distance $d(P, Q)$ of P from a point Q on l is attained at one of the two points of intersection of l with l', and that l' is unique if this minimum distance is less than $\pi/2$.

9. Let $\pi : S^2 \to \mathbb{C}_\infty$ denote the stereographic projection map. Show that the spherical circles on S^2 biject under π with the circles and straight lines on $\mathbb{C}$.

10. Show that any Möbius transformation $T \neq 1$ on $\mathbb{C}_\infty$ has one or two fixed points. Show that the Möbius transformation corresponding (under the stereographic projection map) to a rotation of S^2 through a non-zero angle has exactly two fixed points z_1 and z_2, where $z_2 = -1/\bar{z}_1$. If now T is a Möbius transformation with two fixed points z_1 and z_2 satisfying $z_2 = -1/\bar{z}_1$, prove that **either** T corresponds to a rotation of S^2, **or** one of the fixed points, say z_1, is an *attractive* fixed point, i.e. for $z \neq z_2$, $T^n z \to z_1$ as $n \to \infty$.

11. Prove that Möbius transformations of $\mathbb{C}_\infty$ preserve cross-ratios. If $u, v \in \mathbb{C}$ correspond to points P, Q on S^2, and d denotes the angular distance from P to Q on S^2, show that $-\tan^2 \frac{1}{2}d$ is the cross-ratio of the points $u, v, -1/\bar{u}, -1/\bar{v}$, taken in an appropriate order (which you should specify).

12. Suppose we have a polygonal decomposition of the sphere S^2 or the locally Euclidean torus T by convex geodesic polygons, where each polygon is contained in some hemisphere (for the case of S^2), or is the bijective image of a Euclidean polygon in $\mathbb{R}^2$ under the map $\mathbb{R}^2 \to T$ (for the case of T). If the number of faces (polygons) is F, the number of edges is E and the number of vertices is V, show that $F - E + V = 2$ for the sphere, and $= 0$ for the torus. We denote by F_n the number of faces with precisely n edges, and V_m the number of vertices where precisely m edges meet: show that $\sum_n nF_n = 2E = \sum_m mV_m$.

We suppose that each face has at least three edges, and at least three edges meet at each vertex. If $V_3 = 0$, deduce that $E \geq 2V$. If $F_3 = 0$, deduce that $E \geq 2F$. For the sphere, deduce that $V_3 + F_3 > 0$. For the torus, exhibit a polygonal decomposition with $V_3 = 0 = F_3$.

---

13. For every spherical triangle $\triangle = ABC$, show that $a < b + c$, $b < c + a$, $c < a + b$ and $a + b + c < 2\pi$. Conversely, show that for any three positive numbers a, b, c less than π satisfying the above conditions, we have $\cos(b + c) < \cos a < \cos(b - c)$, and that there is a spherical triangle (unique up to isometries of S^2) with those sides.

14. A spherical triangle $\triangle = ABC$ has vertices given by unit vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in $\mathbb{R}^3$, sides of length a, b, c, and angles α, β, γ (where the side opposite vertex A is of length a and the angle at A is α, etc.). The *polar* triangle $A'B'C'$ is defined by the unit vectors in the directions $\mathbf{B} \times \mathbf{C}$, $\mathbf{C} \times \mathbf{A}$ and $\mathbf{A} \times \mathbf{B}$. Prove that the sides and angles of the polar triangle are $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$, and $\pi - a$, $\pi - b$, $\pi - c$ respectively. Deduce the formula

$$\sin\alpha \sin\beta \cos c = \cos\gamma + \cos\alpha \cos\beta.$$

15. Two spherical triangles $\triangle_1$, $\triangle_2$ on a sphere S^2 are said to be congruent if there is an isometry of S^2 that takes $\triangle_1$ to $\triangle_2$. Show that $\triangle_1$, $\triangle_2$ are congruent if and only if they have equal angles. What other conditions for congruence can you find?

16. With the notation of Question 12, given a polygonal decomposition of S^2 into convex spherical polygons, prove the identity

$$\sum_n (6 - n)F_n = 12 + 2\sum_m (m - 3)V_m.$$

If each face has at least three edges, and at least three edges meet at each vertex, deduce the inequality $3F_3 + 2F_4 + F_5 \geq 12$.

The surface of a football is decomposed into (convex) spherical hexagons and pentagons, with precisely three faces meeting at each vertex. How many pentagons are there? Demonstrate the existence of such a decomposition with each vertex contained in precisely one pentagon.

Note to the reader: You should look at all the questions up to Question 12, and then any further questions you have time for.

Part IB GEOMETRY (Lent 2016): Example Sheet 2

(a.g.kovalev@dpmms.cam.ac.uk)

1. Let $U \subset \mathbb{R}^2$ be an open set equipped with a Riemannian metric $Edu^2 + 2Fdudv + Gdv^2$. For P any point of U, prove that there exists $\lambda > 0$ and an open neighbourhood V of P in U such that

$$(E - \lambda)du^2 + 2Fdudv + (G - \lambda)dv^2$$

is a Riemannian metric on V. [Hint: A real matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite iff $a > 0$ and $ac > b^2$.]

If U is path-connected, we define the distance between two points of U to be the infinum of the lengths of curves joining them; prove that this defines a metric on U. Give an example where this distance is not realized as the length of any curve joining them.

2. We define a Riemannian metric on the unit disc $D \subset \mathbb{C}$ by $(du^2 + dv^2)/(1 - (u^2 + v^2))$. Prove that the diameters (monotonically parametrized) are length minimizing curves for this metric. Defining the distance between two points of D as in Question 1, show that the distances in this metric are bounded, but that the areas are unbounded.

3. We let $V \subset \mathbb{R}^2$ denote the square given by $|u| < 1$ and $|v| < 1$, and define two Riemannian metrics on V given by

$$du^2/(1 - u^2)^2 + dv^2/(1 - v^2)^2, \quad \text{and} \quad du^2/(1 - v^2)^2 + dv^2/(1 - u^2)^2.$$

Prove that there is no isometry between the two spaces, but that an area-preserving diffeomorphism does exist.
[Hint: to prove that an isometry does not exist, show that in one space there are curves of finite length going out to the boundary, whilst in the other space no such curves exist.]

4. Let l denote the hyperbolic line in H given by a semicircle with centre $a \in \mathbb{R}$ and radius $r > 0$. Show that the reflection R_l is given by the formula

$$R_l(z) = a + \frac{r^2}{\bar{z} - a}.$$

5. If a is a point of the upper half-plane, show that the Möbius transformation g given by

$$g(z) = \frac{z - a}{z - \bar{a}}$$

defines an isometry from the upper half-plane model H to the disc model D of the hyperbolic plane, sending a to zero. Deduce that for points z_1, z_2 in the upper half-plane, the hyperbolic distance is given by $\rho(z_1, z_2) = 2\tanh^{-1}\left|(z_1 - z_2)/(z_1 - \overline{z_2})\right|$.

6. Suppose that z_1, z_2 are points in the upper half-plane, and suppose the hyperbolic line through z_1 and z_2 meets the real axis at points z_1^* and z_2^*, where z_1 lies on the hyperbolic line segment $z_1^* z_2$, and where one of z_1^* and z_2^* might be ∞. Show that the hyperbolic distance $\rho(z_1, z_2) = \log r$, where r is the cross-ratio of the four points z_1^*, z_1, z_2, z_2^*, taken in an appropriate order.

7. Let C denote a hyperbolic circle of hyperbolic radius ρ in the upper half-plane model of the hyperbolic plane; show that C is also a Euclidean circle. If C has hyperbolic centre ic, find the radius and centre of C regarded as a Euclidean circle. Show that a hyperbolic circle of hyperbolic radius ρ has hyperbolic area

$$A = 2\pi(\cosh(\rho) - 1).$$

Describe how this function behaves for ρ large; compare the behaviour of the corresponding area functions in Euclidean and spherical geometry.

8. Given two points P and Q in the hyperbolic plane, show that the locus of points equidistant from P and Q is a hyperbolic line, the perpendicular bisector of the hyperbolic line segment from P to Q.

9. Show that any isometry g of the disc model D for the hyperbolic plane is **either** of the form (for some $a \in D$ and $0 \leq \theta < 2\pi$):

$$g(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z},$$

or of the form

$$g(z) = e^{i\theta}\frac{\bar{z}-a}{1-\bar{a}\bar{z}}.$$

10. Prove that a convex hyperbolic n-gon with interior angles $\alpha_1, \ldots, \alpha_n$ has area

$$(n-2)\pi - \sum \alpha_i.$$

Show that for every $n \geq 3$ and every α with $0 < \alpha < (1 - \frac{2}{n})\pi$, there is a regular n-gon all of whose angles are α.

11. Show that two hyperbolic lines have a common perpendicular if and only if they are ultraparallel, and that in this case the perpendicular is unique. Given two ultraparallel hyperbolic lines, prove that the composite of the corresponding reflections has infinite order. [Hint: You may care to take the common perpendicular as a special line.]

12. Fix a point P on the boundary of D, the disc model of the hyperbolic plane. Give a description of the curves in D that are orthogonal to every hyperbolic line that passes through P.

---

13. For arbitrary points z, w in $\mathbb{C}$, prove the identity

$$|1 - \bar{z}w|^2 = |z-w|^2 + (1-|z|^2)(1-|w|^2).$$

Given points z, w in the unit disc model of the hyperbolic plane, prove the identity

$$\sinh^2(\frac{1}{2}\rho(z,w)) = \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}.$$

where ρ denotes the hyperbolic distance.

14. Let $\triangle$ be a hyperbolic triangle, with angles α, β, γ, and sides of length a, b, c (the side of length a being opposite the vertex with angle α, and similarly for b and c). Using the result from Question 13, and the Euclidean cosine rule, prove the hyperbolic cosine rule, namely

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.$$

15. Assuming the hyperbolic cosine rule for hyperbolic triangles, prove the hyperbolic sine rule

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma},$$

[Hint: Reduce this to showing that $(\sinh a \sinh b)^2 - (\cosh a \cosh b - \cosh c)^2$ is symmetric in a, b and c. For a slicker proof of the hyperbolic cosine and sine rules, via the hyperboloid model of hyperbolic space, consult P.M.H. Wilson *Curved Spaces*, § 5.7.]

16. Let l be a hyperbolic line and P a point on l. Show that there is a unique hyperbolic line l' through P making an angle α with l (in a given sense). If α, β are positive numbers with $\alpha + \beta < \pi$, show that there exists a hyperbolic triangle (one vertex at infinity) with angles 0, α and β. For any positive numbers α, β, γ, with $\alpha + \beta + \gamma < \pi$, show that there exists a hyperbolic triangle with these angles. [Hint: For the last part, you may need a continuity argument.]

Note to the reader: You should look at all the questions up to Question 12, and then any further questions you have time for.

Part IB GEOMETRY (Lent 2016): Example Sheet 3

(a.g.kovalev@dpmms.cam.ac.uk)

1. Let V be the open subset $\{0 < u < \pi,\ 0 < v < 2\pi\}$, and $\sigma : V \to S^2$ be given by

$$\sigma(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

Prove that σ defines a smooth parametrization on a certain open subset of S^2. [*You may assume* $\cos^{-1}$ *is continuous on* $(-1, 1)$, *and* $\tan^{-1}$, $\cot^{-1}$ *are continuous on* $(-\infty, \infty)$.]

2. Show that the tangent space to S^2 at a point $P = (x, y, z) \in S^2$ is the plane normal to the vector $\overrightarrow{OP}$, where O denotes the origin.

3. Show the stereographic projection map $\pi : S \setminus \{N\} \to \mathbb{C}$, where N denotes the north pole, defines a chart. Check that the spherical metric on $S \setminus \{N\}$ corresponds under π to the Riemannian metric on $\mathbb{C}$ given by $4(dx^2 + dy^2)/(1 + x^2 + y^2)^2$.

4. For each map $\sigma : U \to \mathbb{R}^3$, find the Riemannian metric on U induced by σ. Sketch the image of σ in $\mathbb{R}^3$.
(a) $U = \{(u, v) \in \mathbb{R}^2 : u > v\}$, $\sigma(u, v) = (u + v,\ 2uv,\ u^2 + v^2)$.
(b) $U = \{(r, z) \in \mathbb{R}^2 : r > 0\}$, $\sigma(r, z) = (r \cos z,\ r \sin z,\ z)$.

5. Let T denote the embedded torus in $\mathbb{R}^3$ obtained by revolving around the z-axis the circle $(x - 2)^2 + z^2 = 1$ in the xz-plane. Using the formal definition of area in terms of a parametrization, calculate the surface area of T.

6. If one places S^2 inside a (vertical) circular cylinder of radius one, prove that the radial (horizontal) projection map from S^2 to the cylinder preserves areas (this is usually known as *Archimedes Theorem*). Deduce the existence of an atlas on S^2, for which the charts all preserve areas and the transition functions have derivatives with determinant one.

7. Using the geodesic equations, show directly that the geodesics in the hyperbolic plane are hyperbolic lines parametrized with constant speed. [Hint: In the upper half-plane model, prove that a geodesic curve between any two points of the positive imaginary axis L^+ is of the form claimed.]

8. For $a > 0$, let $S \subset \mathbb{R}^3$ be the circular half-cone defined by $z^2 = a(x^2 + y^2)$, $z > 0$, considered as an embedded surface. Show that S minus a ray through the origin is isometric to a suitable region in the plane. [Intuitively: you can glue a piece of paper to form a cone, without any crumpling of the paper.] When $a = 3$, give an explicit formula for the geodesics on S and show that no geodesic intersects itself. When $a > 3$, show that there are geodesics (of infinite length) which intersect themselves.

9. For a surface of revolution S, corresponding to an embedded curve $\eta : (a, b) \to \mathbb{R}^3$ given by $\eta(u) = (f(u), 0, g(u))$, where η' is never zero, η is a homeomorphism onto its image, and $f(u)$ is always positive, prove that the Gaussian curvature K is given by the formula

$$K = \frac{(f'g'' - f''g')g'}{f((f')^2 + (g')^2)^2}.$$

In the case when η is parametrized in such a way that $\|\eta'\| = 1$, prove that K is given by the formula $K = -f''/f$. Verify that the unit sphere has constant curvature 1.

10. Using the results from the previous question, calculate the Gaussian curvature K for the hyperboloid of one sheet $x^2 + y^2 = z^2 + 1$, and the hyperboloid of two sheets $x^2 + y^2 = z^2 - 1$. Describe the qualitative properties of the curvature in these cases (sign and behavior near infinity), and explain what you find using pictures of these surfaces.

For the embedded torus, as defined in Question 4, identify those points at which $K = 0$, $K > 0$ and $K < 0$. Verify the global Gauss–Bonnet theorem on the embedded torus.

11. Suppose we have a Riemannian metric of the form $|dz|^2/h(r)^2$ on some open disc $D(0;\delta)$ centred at the origin in $\mathbb{C}$ (possibly all of $\mathbb{C}$), where $r = |z|$ and $h(r) > 0$ for all $r < \delta$. Show that the curvature K of this metric is given by the formula $K = hh'' - (h')^2 + hh'/r$.

12. Show that the embedded surface S with equation $x^2 + y^2 + c^2z^2 = 1$, where $c > 0$, is homeomorphic to the sphere. Deduce from the Gauss–Bonnet theorem that

$$\int_0^1 (1 + (c^2 - 1)u^2)^{-3/2} du = c^{-1}.$$

Can you find a direct verification of this formula?

---

13. Given a smooth curve $\Gamma : [0,1] \to S$ on an abstract surface S with a Riemannian metric, show that the length l is unchanged under reparametrizations of the form $f : [0,1] \to [0,1]$, with $f'(t) > 0$ for all $t \in [0,1]$. Prove that if $\Gamma'(t) \neq 0$ for all t, then Γ can be reparametrized to a curve with constant speed.

14. Show that Mercator's parametrization of the sphere (minus poles)

$$\sigma(u,v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u)$$

determines a chart (on the complement of a longitude) which preserves angles and sends meridians and parallels on the sphere to straight lines in the plane.

15. Let S be an embedded surface in $\mathbb{R}^3$ which is closed and bounded. By considering the smallest closed ball centred on the origin which contains S, or otherwise, show that the Gaussian curvature must be strictly positive at some point of S. Deduce that the locally Euclidean metric on the torus T cannot be realized as the first fundamental form of a smooth embedding of T in $\mathbb{R}^3$.

16. Show that the surface obtained by attaching 2 handles to a sphere (i.e. the surface of a 'doughnut with 2 holes') may be obtained topologically by suitably identifying the sides of a regular octagon. Indicate briefly how to extend your argument to show that a 'sphere with g handles' Σ_g may be obtained topologically by suitably identifying the sides of a regular $4g$-gon.

Show that Σ_g $(g > 1)$ may be given the structure of an abstract surface with a Riemannian metric, in such a way that it is locally isometric to the hyperbolic plane. [For this question, you will need the result from Q10 on Example Sheet 2.]

Note to the reader: You should look at all the questions up to Question 12, and then any further questions you have time for.