Part IB — Geometry
Definitions

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Parts of Analysis II will be found useful for this course.

Groups of rigid motions of Euclidean space. Rotation and reflection groups in two and three dimensions. Lengths of curves. [2]

Spherical geometry: spherical lines, spherical triangles and the Gauss-Bonnet theorem. Stereographic projection and Möbius transformations. [3]

Triangulations of the sphere and the torus, Euler number. [1]

Riemannian metrics on open subsets of the plane. The hyperbolic plane. Poincaré models and their metrics. The isometry group. Hyperbolic triangles and the Gauss-Bonnet theorem. The hyperboloid model. [4]

Embedded surfaces in $\mathbb{R}^3$. The first fundamental form. Length and area. Examples. [1]

Length and energy. Geodesics for general Riemannian metrics as stationary points of the energy. First variation of the energy and geodesics as solutions of the corresponding Euler-Lagrange equations. Geodesic polar coordinates (informal proof of existence). Surfaces of revolution. [2]

The second fundamental form and Gaussian curvature. For metrics of the form $du^2 + G(u, v)dv^2$, expression of the curvature as $\sqrt{G_{uu}}/\sqrt{G}$. Abstract smooth surfaces and isometries. Euler numbers and statement of Gauss-Bonnet theorem, examples and applications. [3]

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0 Introduction

1 Euclidean geometry

1.1 Isometries of the Euclidean plane

Definition ((Standard) inner product)**.** The *(standard) inner product* on $\mathbb{R}^n$ is defined by

$$(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

Definition (Euclidean Norm)**.** The *Euclidean norm* of $\mathbf{x} \in \mathbb{R}^n$ is

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}.$$

This defines a metric on $\mathbb{R}^n$ by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

Definition (Isometry)**.** A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is an *isometry* of $\mathbb{R}^n$ if

$$d(f(\mathbf{x}), f(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Definition (Orthogonal matrix)**.** An $n \times n$ matrix A is *orthogonal* if $AA^T = A^T A = I$. The group of all orthogonal matrices is the orthogonal group $\mathrm{O}(n)$.

Definition (Isometry group)**.** The *isometry group* $\mathrm{Isom}(\mathbb{R}^n)$ is the group of all isometries of $\mathbb{R}^n$, which is a group by composition.

Definition (Special orthogonal group)**.** The *special orthogonal group* is the group

$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \det A = 1\}.$$

Definition (Orientation)**.** An *orientation* of a vector space is an equivalence class of bases — let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ and $\mathbf{v}'_1, \cdots, \mathbf{v}'_n$ be two bases and A be the change of basis matrix. We say the two bases are equivalent iff $\det A > 0$. This is an equivalence relation on the bases, and the equivalence classes are the orientations.

Definition (Orientation-preserving isometry)**.** An isometry $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ is *orientation-preserving* if $\det A = 1$. Otherwise, if $\det A = -1$, we say it is *orientation-reversing*.

1.2 Curves in $\mathbb{R}^n$

Definition (Curve)**.** A *curve* Γ in $\mathbb{R}^n$ is a continuous map $\Gamma : [a, b] \to \mathbb{R}^n$.

Definition (Length of curve)**.** The length of a curve $\Gamma : [a, b] \to \mathbb{R}^n$ is

$$\ell = \sup_{\mathcal{D}} S_{\mathcal{D}},$$

if the supremum exists.

2 Spherical geometry

Notation. We write $S = S^2 \subseteq \mathbb{R}^3$ for the unit sphere. We write $O = \mathbf{0}$ for the origin, which is the center of the sphere (and not on the sphere).

Definition (Great circle)**.** A *great circle* (in S^2) is $S^2 \cap$ (a plane through O). We also call these *(spherical) lines*.

Definition (Distance on a sphere)**.** Given $P, Q \in S$, the *distance* $d(P, Q)$ is the shorter of the two (spherical) line segments (i.e. arcs) PQ along the respective great circle. When P and Q are antipodal, there are infinitely many line segments between them of the same length, and the distance is π.

2.1 Triangles on a sphere

2.2 Möbius geometry

3 Triangulations and the Euler number

Definition ((Euclidean) torus)**.** The *(Euclidean) torus* is the set $\mathbb{R}^2/\mathbb{Z}^2$ of equivalence classes of $(x, y) \in \mathbb{R}^2$ under the equivalence relation

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 - x_2, y_1 - y_2 \in \mathbb{Z}.$$

Definition (Topological triangle)**.** A *topological triangle* on $X = S^2$ or T (or any metric space X) is a subset $R \subseteq X$ equipped with a homeomorphism $R \to \Delta$, where Δ is a closed Euclidean triangle in $\mathbb{R}^2$.

Definition (Topological triangulation)**.** A *topological triangulation* τ of a metric space X is a finite collection of topological triangles of X which cover X such that

(i) For every pair of triangles in τ, either they are disjoint, or they meet in exactly one edge, or meet at exactly one vertex.

(ii) Each edge belongs to exactly two triangles.

Definition (Euler number)**.** The *Euler number* of a triangulation $e = e(X, \tau)$ is

$$e = F - E + V,$$

where F is the number of triangles; E is the number of edges; and V is the number of vertices.

Definition (Geodesic triangle)**.** A *geodesic triangle* is a triangle whose sides are geodesics, i.e. paths of shortest distance between two points.

4 Hyperbolic geometry

4.1 Review of derivatives and chain rule

Definition (Smooth function)**.** Let $U \subseteq \mathbb{R}^n$ be open, and $f = (f_1, \cdots, f_m) : U \to \mathbb{R}^m$. We say f is *smooth* (or C^∞) if each f_i has continuous partial derivatives of each order. In particular, a C^∞ map is differentiable, with continuous first-order partial derivatives.

Definition (Derivative)**.** The *derivative* for a function $f : U \to \mathbb{R}^m$ at a point $a \in U$ is a linear map $\mathrm{d}f_a : \mathbb{R}^n \to \mathbb{R}^m$ (also written as $\mathrm{D}f(a)$ or $f'(a)$) such that

$$\lim_{h \to 0} \frac{\|f(a + h) - f(a) - \mathrm{d}f_a \cdot h\|}{\|h\|} \to 0,$$

where $h \in \mathbb{R}^n$.

If $m = 1$, then $\mathrm{d}f_a$ is expressed as $\left(\frac{\partial f}{\partial x_a}(a), \cdots, \frac{\partial f}{\partial x_n}(a)\right)$, and the linear map is given by

$$(h_1, \cdots, h_n) \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a) h_i,$$

i.e. the dot product. For a general m, this vector becomes a matrix. The *Jacobian matrix* is

$$J(f)_a = \left(\frac{\partial f_i}{\partial x_j}(a)\right),$$

with the linear map given by matrix multiplication, namely

$$h \mapsto J(f)_a \cdot h.$$

4.2 Riemannian metrics

Definition (Riemannian metric)**.** We use coordinates $(u, v) \in \mathbb{R}^2$. We let $V \subseteq \mathbb{R}^2$ be open. Then a Riemannian metric on V is defined by giving C^∞ functions $E, F, G : V \to \mathbb{R}$ such that

$$\begin{pmatrix} E(P) & F(P) \\ F(P) & G(P) \end{pmatrix}$$

is a positive definite definite matrix for all $P \in V$.

Alternatively, this is a smooth function that gives a 2×2 symmetric positive definite matrix, i.e. inner product $\langle\,\cdot\,,\,\cdot\,\rangle_P$, for each point in V. By definition, if $\mathbf{e}_1, \mathbf{e}_2$ are the standard basis, then

$$\begin{aligned} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle_P &= E(P) \\ \langle \mathbf{e}_1, \mathbf{e}_2 \rangle_P &= F(P) \\ \langle \mathbf{e}_2, \mathbf{e}_2 \rangle_P &= G(P). \end{aligned}$$

Definition (Length)**.** The *length* of a smooth curve $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to V$ is defined as

$$\int_0^1 \left(E\dot{\gamma}_1^2 + 2F\dot{\gamma}_1\dot{\gamma}_2 + G\dot{\gamma}_2{}^2\right)^{\frac{1}{2}}\ \mathrm{d}t,$$

where $E = E(\gamma_1(t), \gamma_2(t))$ etc. We can also write this as

$$\int_0^1 \langle \dot{\gamma}, \dot{\gamma}\rangle_{\gamma(t)}^{\frac{1}{2}}\;\mathrm{d}t.$$

Definition (Area). The *area* of a region $W \subseteq V$ is defined as

$$\int_W (EG - F^2)^{\frac{1}{2}}\;\mathrm{d}u\;\mathrm{d}v$$

when this integral exists.

Definition (Isometry). Let $V, \tilde{V} \subseteq \mathbb{R}^2$ be open sets endowed with Riemannian metrics, denoted as $\langle\,\cdot\,,\,\cdot\,\rangle_P$ and $\langle\,\cdot\,,\,\cdot\,\rangle_Q^\sim$ for $P \in V, Q \in \tilde{V}$ respectively.

A diffeomorphism (i.e. C^∞ map with C^∞ inverse) $\varphi : V \to \tilde{V}$ is an *isometry* if for every $P \in V$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we get

$$\langle \mathbf{x}, \mathbf{y}\rangle_P = \langle \mathrm{d}\varphi_P(\mathbf{x}), \mathrm{d}\varphi_P(\mathbf{y})\rangle_{\varphi(P)}^\sim.$$

4.3 Two models for the hyperbolic plane

Definition (Poincaré disk model). The *(Poincaré) disk model* for the hyperbolic plane is given by the unit disk

$$D \subseteq \mathbb{C} \cong \mathbb{R}^2, \quad D = \{\zeta \in \mathbb{C} : |\zeta| < 1\},$$

and a Riemannian metric on this disk given by

$$\frac{4(\mathrm{d}u^2 + \mathrm{d}v^2)}{(1 - u^2 - v^2)^2} = \frac{4|\mathrm{d}\zeta|^2}{(1 - |\zeta|^2)^2}, \tag{$*$}$$

where $\zeta = u + iv$.

Definition (Upper half-plane). The *upper half-plane* is

$$H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

Definition (Upper half-plane model). The *upper half-plane* model of the hyperbolic plane is the upper half-plane H with the Riemannian metric

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

Definition (Hyperbolic lines). *Hyperbolic lines* in H are vertical half-lines or semicircles ending in $\mathbb{R}$.

Definition (Hyperbolic distance). For points $z_1, z_2 \in H$, the *hyperbolic distance* $\rho(z_1, z_2)$ is the length of the segment $[z_1, z_2] \subseteq \ell$ of the hyperbolic line through z_1, z_2 (parametrized monotonically).

4.4 Geometry of the hyperbolic disk

Definition (Hyperbolic reflection). The map $R : z \in H \mapsto -\bar{z} \in H$ is the *(hyperbolic) reflection in* L^+. More generally, given any hyperbolic line ℓ, let T be the isometry that sends ℓ to L^+. Then the *(hyperbolic) reflection in* ℓ is

$$R_\ell = T^{-1}RT$$

4.5 Hyperbolic triangles

Definition (Hyperbolic triangle)**.** A *hyperbolic triangle* ABC is the region determined by three hyperbolic line segments AB, BC and CA, including extreme cases where some vertices A, B, C are allowed to be "at infinity". More precisely, in the half-plane model, we allow them to lie in $\mathbb{R} \cup \{\infty\}$; in the disk model we allow them to lie on the unit circle $|z| = 1$.

Definition (Parallel lines)**.** We use the disk model of the hyperbolic plane. Two hyperbolic lines are *parallel* iff they meet only at the boundary of the disk (at $|z| = 1$).

Definition (Ultraparallel lines)**.** Two hyperbolic lines are *ultraparallel* if they don't meet anywhere in $\{|z| \leq 1\}$.

4.6 Hyperboloid model

Definition (Lorentzian inner product)**.** The *Lorentzian inner product* on $\mathbb{R}^3$ has the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

5 Smooth embedded surfaces (in $\mathbb{R}^3$)

5.1 Smooth embedded surfaces

Definition (Smooth embedded surface)**.** A set $S \subseteq \mathbb{R}^3$ is a *(parametrized) smooth embedded surface* if every point $P \in S$ has an open neighbourhood $U \subseteq S$ (with the subspace topology on $S \subseteq \mathbb{R}^3$) and a map $\sigma : V \to U$ from an open $V \subseteq \mathbb{R}^2$ to U such that if we write $\sigma(u, v) = (x(u, v), y(u, v), z(u, v))$, then

(i) σ is a homeomorphism (i.e. a bijection with continuous inverse)

(ii) σ is C^∞ (smooth) on V (i.e. has continuous partial derivatives of all orders).

(iii) For all $Q \in V$, the partial derivatives $\sigma_u(Q)$ and $\sigma_v(Q)$ are linearly independent.

Definition (Smooth coordinates)**.** We say (u, v) are *smooth coordinates* on $U \subseteq S$.

Definition (Tangent space)**.** The subspace of $\mathbb{R}^3$ spanned by $\sigma_u(Q), \sigma_v(Q)$ is the *tangent space* $T_P S$ to S at $P = \sigma(Q)$.

Definition (Smooth parametrisation)**.** The function σ is a *smooth parametrisation* of $U \subseteq S$.

Definition (Chart)**.** The function $\sigma^{-1} : U \to V$ is a *chart* of U.

Definition (Unit normal)**.** The *unit normal* to S at $Q \in S$ is

$$N = N_Q = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|},$$

which is well-defined up to a sign.

Definition (Chart)**.** Let $S \subseteq \mathbb{R}^3$ be an embedded surface. The map $\theta = \sigma^{-1} : U \subseteq S \to V \subseteq \mathbb{R}^2$ is a *chart.*

Definition (First fundamental form)**.** If $S \subseteq \mathbb{R}^3$ is an embedded surface, then each $T_Q S$ for $Q \in S$ has an inner product from $\mathbb{R}^3$, i.e. we have a family of inner products, one for each point. We call this family the *first fundamental form.*

Definition (Length and energy of curve)**.** Given a smooth curve $\Gamma : [a, b] \to S \subseteq \mathbb{R}^3$, the *length* of γ is

$$\mathrm{length}(\Gamma) = \int_a^b \|\Gamma'(t)\| \;\mathrm{d}t.$$

The *energy* of the curve is

$$\mathrm{energy}(\Gamma) = \int_a^b \|\Gamma'(t)\|^2 \;\mathrm{d}t.$$

Definition (Area)**.** Given a smooth C^∞ parametrization $\sigma : V \to U \subseteq S \subseteq \mathbb{R}^3$, and a region $T \subseteq U$, we define the *area* of T to be

$$\mathrm{area}(T) = \int_{\theta(T)} \sqrt{EG - F^2} \;\mathrm{d}u\;\mathrm{d}v,$$

whenever the integral exists (where $\theta = \sigma^{-1}$ is a chart).

5.2 Geodesics

Definition (Geodesic)**.** Let $V \subseteq \mathbb{R}^2_{u,v}$ be open, and $E\ \mathrm{d}u^2 + 2F\ \mathrm{d}u\ \mathrm{d}v + G\ \mathrm{d}v^2$ be a Riemannian metric on V. We let

$$\gamma = (\gamma_1, \gamma_2) : [a, b] \to V$$

be a smooth curve. We say γ is a *geodesic* with respect to the Riemannian metric if it satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{\gamma}_1 + F\dot{\gamma}_2) &= \frac{1}{2}(E_u\dot{\gamma}_1^2 + 2F_u\dot{\gamma}_1\dot{\gamma}_2 + G_u\dot{\gamma}_2^2)\\ \frac{\mathrm{d}}{\mathrm{d}t}(F\dot{\gamma}_1 + G\dot{\gamma}_2) &= \frac{1}{2}(E_v\dot{\gamma}_1^2 + 2F_v\dot{\gamma}_1\dot{\gamma}_2 + G_v\dot{\gamma}_2^2)\end{aligned}$$

for all $t \in [a, b]$. These equations are known as the *geodesic ODEs*.

Definition (Proper variation)**.** Let $\gamma : [a, b] \to V$ be a smooth curve, and let $\gamma(a) = p$ and $\gamma(b) = q$. A *proper variation* of γ is a C^∞ map

$$h : [a, b] \times (-\varepsilon, \varepsilon) \subseteq \mathbb{R}^2 \to V$$

such that

$$h(t, 0) = \gamma(t) \text{ for all } t \in [a, b],$$

and

$$h(a, \tau) = p, \quad h(b, \tau) = q \text{ for all } |\tau| < \varepsilon,$$

and that

$$\gamma_\tau = h(\,\cdot\,, \tau) : [a, b] \to V$$

is a C^∞ curve for all fixed $\tau \in (-\varepsilon, \varepsilon)$.

Definition (Geodesic on smooth embedded surface)**.** Let $S \subseteq \mathbb{R}^3$ be an embedded surface. Let $\Gamma : [a, b] \to S$ be a smooth curve in S, and suppose there is a parametrization $\sigma : V \to U \subseteq S$ such that $\operatorname{im}\Gamma \subseteq U$. We let $\theta = \sigma^{-1}$ be the corresponding chart.

We define a new curve in V by

$$\gamma = \theta \circ \Gamma : [a, b] \to V.$$

Then we say Γ is a *geodesic* on S if and only if γ is a geodesic with respect to the induced Riemannian metric.

For a general $\Gamma : [a, b] \to V$, we say Γ is a *geodesic* if for each point $t_0 \in [a, b]$, there is a neighbourhood $\tilde{V}$ of t_0 such that $\operatorname{im}\Gamma|_{\tilde{V}}$ lies in the domain of some chart, and $\Gamma|_{\tilde{V}}$ is a geodesic in the previous sense.

Definition (Atlas)**.** An *atlas* is a collection of charts covering the whole surface.

5.3 Surfaces of revolution

Definition (Parallels)**.** On a surface of revolution, *parallels* are curves of the form

$$\gamma(t) = \sigma(u_0, t) \text{ for fixed } u_0.$$

Meridians are curves of the form

$$\gamma(t) = \sigma(t, v_0) \text{ for fixed } v_0.$$

5.4 Gaussian curvature

Definition (Curvature of curve)**.** We let $\eta : [0, \ell] \to R^2$ be a curve parametrized with unit speed, i.e. $\|\eta'\| = 1$. The *curvature* κ at the point $\eta(s)$ is determined by

$$\eta'' = \kappa \mathbf{n},$$

where $\mathbf{n}$ is the unit normal, chosen so that κ is non-negative.

Definition (Second fundamental form)**.** The *second fundamental form* on V with $\sigma : V \to U \subseteq S$ for S is

$$L\ \mathrm{d}u^2 + 2M\ \mathrm{d}u\ \mathrm{d}v + N\ \mathrm{d}v^2,$$

where

$$\begin{aligned} L &= \sigma_{uu} \cdot \mathbf{N} \\ M &= \sigma_{uv} \cdot \mathbf{N} \\ N &= \sigma_{vv} \cdot \mathbf{N}. \end{aligned}$$

Definition (Gaussian curvature)**.** The *Gaussian curvature* K of a surface of S at $P \in S$ is the ratio of the determinants of the two fundamental forms, i.e.

$$K = \frac{LN - M^2}{EG - F^2}.$$

This is valid since the first fundamental form is positive-definite and in particular has non-zero derivative.

6 Abstract smooth surfaces

Definition (Abstract smooth surface)**.** An *abstract smooth surface* S is a metric space (or Hausdorff (and second-countable) topological space) equipped with homeomorphisms $\theta_i : \mathcal{U}_i \to V_i$, where $\mathcal{U}_i \subseteq S$ and $V_i \subseteq \mathbb{R}^2$ are open sets such that

(i) $S = \bigcup_i \mathcal{U}_i$

(ii) For any i, j, the transition map
$$\phi_{ij} = \theta_j \circ \theta_i^{-1} : \theta_j(\mathcal{U}_i \cap \mathcal{U}_j) \to \theta_i(\mathcal{U}_i \cap \mathcal{U}_j)$$
is a diffeomorphism. Note that $\theta_j(\mathcal{U}_i \cap \mathcal{U}_j)$ and $\theta_i(\mathcal{U}_i \cap \mathcal{U}_j)$ are open sets in $\mathbb{R}^2$. So it makes sense to talk about whether the function is a diffeomorphism.

Definition (Riemannian metric on abstract surface)**.** A *Riemannian metric* on an abstract surface is given by Riemannian metrics on each $V_i = \theta_i(\mathcal{U}_i)$ subject to the compatibility condition that for all i, j, the transition map ϕ_{ij} is an isometry, i.e.
$$\langle d\varphi_P(\mathbf{a}), \mathrm{d}\varphi_P(\mathbf{b})\rangle_{\varphi(P)} = \langle \mathbf{a}, \mathbf{b}\rangle_P$$
Note that on the left, we are computing the Riemannian metric on V_i, while on the left, we are computing it on V_j.