Part IB — Fluid Dynamics
Theorems with proof

Based on lectures by P. F. Linden
Notes taken by Dexter Chua

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Parallel viscous flow
Plane Couette flow, dynamic viscosity. Momentum equation and boundary conditions. Steady flows including Poiseuille flow in a channel. Unsteady flows, kinematic viscosity, brief description of viscous boundary layers (skin depth). [3]

Kinematics
Material time derivative. Conservation of mass and the kinematic boundary condition. Incompressibility; streamfunction for two-dimensional flow. Streamlines and path lines. [2]

Dynamics
Statement of Navier-Stokes momentum equation. Reynolds number. Stagnation-point flow; discussion of viscous boundary layer and pressure field. Conservation of momentum; Euler momentum equation. Bernoulli's equation.

Vorticity, vorticity equation, vortex line stretching, irrotational flow remains irrotational. [4]

Potential flows
Velocity potential; Laplace's equation, examples of solutions in spherical and cylindrical geometry by separation of variables. Translating sphere. Lift on a cylinder with circulation.

Expression for pressure in time-dependent potential flows with potential forces. Oscillations in a manometer and of a bubble. [3]

Geophysical flows
Linear water waves: dispersion relation, deep and shallow water, standing waves in a container, Rayleigh-Taylor instability.

Euler equations in a rotating frame. Steady geostrophic flow, pressure as streamfunction. Motion in a shallow layer, hydrostatic assumption, modified continuity equation. Conservation of potential vorticity, Rossby radius of deformation. [4]

Contents

0 Introduction

1 Parallel viscous flow

1.1 Preliminaries

1.2 Stress

Law. For a Newtonian fluid, we have

$$\tau_s \propto \frac{U}{h}.$$

1.3 Steady parallel viscous flow

1.4 Derived properties of a flow

1.5 More examples

2 Kinematics

2.1 Material time derivative

2.2 Conservation of mass

2.3 Kinematic boundary conditions

2.4 Streamfunction for incompressible flow

3 Dynamics

3.1 Navier-Stokes equations

Law (Navier-Stokes equation)**.**

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}.$$

3.2 Pressure

3.3 Reynolds number

3.4 A case study: stagnation point flow ($\mathbf{u} = \mathbf{0}$)

3.5 Momentum equation for inviscid ($\nu = 0$) incompressible fluid

Proposition (Euler momentum equation)**.**

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mathbf{f}.$$

Proposition (Momentum integral for steady flow)**.**

$$\int_{\partial \mathcal{D}} (\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n} + \chi \mathbf{n})\; \mathrm{d}S = 0.$$

Proposition (Bernoulli's equation)**.**

$$\frac{1}{2}\rho \frac{\partial |\mathbf{u}|^2}{\partial t} = -\mathbf{u} \cdot \nabla \left(\frac{1}{2}\rho |\mathbf{u}|^2 + p + \chi\right).$$

3.6 Linear flows

3.7 Vorticity equation

Proposition (Vorticity equation)**.**

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

4 Inviscid irrotational flow

4.1 Three-dimensional potential flows

4.2 Potential flow in two dimensions

4.3 Time dependent potential flows

5 Water waves

5.1 Dimensional analysis

5.2 Equation and boundary conditions

5.3 Two-dimensional waves (straight crested waves)

5.4 Group velocity

5.5 Rayleigh-Taylor instability

6 Fluid dynamics on a rotating frame

6.1 Equations of motion in a rotating frame

Proposition (Euler's equation in a rotating frame)**.**

$$\frac{\partial \mathbf{u}}{\partial t} + 2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}.$$

6.2 Shallow water equations

6.3 Geostrophic balance