

# Part IB — Fluid Dynamics

## Theorems

Based on lectures by P. F. Linden

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

### **Parallel viscous flow**

Plane Couette flow, dynamic viscosity. Momentum equation and boundary conditions. Steady flows including Poiseuille flow in a channel. Unsteady flows, kinematic viscosity, brief description of viscous boundary layers (skin depth). [3]

### **Kinematics**

Material time derivative. Conservation of mass and the kinematic boundary condition. Incompressibility; streamfunction for two-dimensional flow. Streamlines and path lines. [2]

### **Dynamics**

Statement of Navier-Stokes momentum equation. Reynolds number. Stagnation-point flow; discussion of viscous boundary layer and pressure field. Conservation of momentum; Euler momentum equation. Bernoulli's equation.

Vorticity, vorticity equation, vortex line stretching, irrotational flow remains irrotational. [4]

### **Potential flows**

Velocity potential; Laplace's equation, examples of solutions in spherical and cylindrical geometry by separation of variables. Translating sphere. Lift on a cylinder with circulation.

Expression for pressure in time-dependent potential flows with potential forces. Oscillations in a manometer and of a bubble. [3]

### **Geophysical flows**

Linear water waves: dispersion relation, deep and shallow water, standing waves in a container, Rayleigh-Taylor instability.

Euler equations in a rotating frame. Steady geostrophic flow, pressure as streamfunction. Motion in a shallow layer, hydrostatic assumption, modified continuity equation. Conservation of potential vorticity, Rossby radius of deformation. [4]

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## **0 Introduction**

## 1 Parallel viscous flow

### 1.1 Preliminaries

### 1.2 Stress

**Law.** For a Newtonian fluid, we have

$$\tau_s \propto \frac{U}{h}.$$

### 1.3 Steady parallel viscous flow

### 1.4 Derived properties of a flow

### 1.5 More examples

## **2 Kinematics**

**2.1 Material time derivative**

**2.2 Conservation of mass**

**2.3 Kinematic boundary conditions**

**2.4 Streamfunction for incompressible flow**

### 3 Dynamics

#### 3.1 Navier-Stokes equations

**Law** (Navier-Stokes equation).

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}.$$

#### 3.2 Pressure

#### 3.3 Reynolds number

#### 3.4 A case study: stagnation point flow ( $\mathbf{u} = 0$ )

#### 3.5 Momentum equation for inviscid ( $\nu = 0$ ) incompressible fluid

**Proposition** (Euler momentum equation).

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{f}.$$

**Proposition** (Momentum integral for steady flow).

$$\int_{\partial \mathcal{D}} (\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p \mathbf{n} + \chi \mathbf{n}) \, dS = 0.$$

**Proposition** (Bernoulli's equation).

$$\frac{1}{2} \rho \frac{\partial |\mathbf{u}|^2}{\partial t} = -\mathbf{u} \cdot \nabla \left( \frac{1}{2} \rho |\mathbf{u}|^2 + p + \chi \right).$$

#### 3.6 Linear flows

#### 3.7 Vorticity equation

**Proposition** (Vorticity equation).

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

## 4 Inviscid irrotational flow

### 4.1 Three-dimensional potential flows

### 4.2 Potential flow in two dimensions

### 4.3 Time dependent potential flows

## 5 Water waves

### 5.1 Dimensional analysis

### 5.2 Equation and boundary conditions

### 5.3 Two-dimensional waves (straight crested waves)

### 5.4 Group velocity

### 5.5 Rayleigh-Taylor instability

## 6 Fluid dynamics on a rotating frame

### 6.1 Equations of motion in a rotating frame

**Proposition** (Euler's equation in a rotating frame).

$$\frac{\partial \mathbf{u}}{\partial t} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{g}.$$

### 6.2 Shallow water equations

### 6.3 Geostrophic balance