Fluid Dynamics IB
Lent 2016
Professor P F Linden

Example Sheet 1

Note to students and supervisors: Every answer should include a relevant sketch.

1. By considering the forces acting on a slab of incompressible, viscous fluid undergoing an unsteady parallel shear flow $\mathbf{u} = (u(y,t), 0, 0)$ acted on by a body force (force per unit volume) $\mathbf{f} = (f_x, f_y, 0)$, in Cartesian coordinates (x, y, z), show that

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} + f_x,$$
$$0 = -\frac{\partial p}{\partial y} + f_y.$$

2. A film of viscous fluid of uniform thickness h flows steadily under the influence of gravity down a rigid vertical wall. Assume that the surrounding air exerts no stress on the fluid. Calculate the velocity profile and find the volume flux (per unit width) of fluid down the wall.

3. A long, horizontal, two-dimensional container of depth h, filled with viscous fluid, has rigid, stationary bottom and end walls and a rigid top wall that moves with velocity $(U, 0)$ in Cartesian coordinates (x, y), where x is horizontal and U is constant. Assume that the fluid flow far from the end walls is parallel and steady with components $(u(y), 0)$. Determine $u(y)$ and hence determine the tangential force per unit area exerted by the fluid on each of the top and bottom walls.

[*Hint: No penetration through the end walls demands that the volume flux across any vertical cross-section of the flow is zero, which determines the horizontal pressure gradient.*]

4. A two-dimensional, semi-infinite layer of viscous fluid lies above a rigid boundary at $y = 0$ that is oscillating in its own plane with velocity $(U_0 \cos \omega t, 0)$. Assume that there is no pressure gradient and that the fluid flows parallel to the boundary with velocity $(u(y,t), 0)$. By writing $u(y,t) = Re[U_0 f(y) e^{i\omega t}]$ or otherwise, show that

$$f(y) = \exp\left[-(1+i)\sqrt{\frac{\omega}{2\nu}}y\right]$$

and hence that the velocity decays within a characteristic distance of $\sqrt{\nu/\omega}$ from the boundary.

Calculate the shear stress on the boundary and hence calculate the mean rate of doing work per unit area of the boundary.

5. An infinite horizontal layer of viscous fluid of depth h is initially stationary and has a rigid, stationary upper boundary while its lower, rigid boundary is set into parallel motion with constant speed U at time $t = 0$. Write down the equation, the initial condition and the boundary conditions satisfied by the subsequent flow $(u(y,t), 0)$. What is the steady flow $u_\infty(y)$ that is established after a long time? By writing $u(y,t) = u_\infty(y) - \hat{u}(y,t)$ and using separation of variables, determine a series solution for the transient flow $\hat{u}$.

Show that the shear stress exerted by the fluid on the boundary at $y = 0$ is divergent as $t \to 0^+$ but that it is subsequently finite and tends to $-\mu U/h$ as $t \to \infty$.

6. Consider the two-dimensional flow $u = 1/(1+t)$, $v = 1$ in $t > 0$. Find and sketch
(i) the streamline at $t = 0$ which passes through the point $(1,1)$,
(ii) the path of a fluid particle which is released from $(1,1)$ at $t = 0$,

7. A two-dimensional flow is represented by a streamfunction $\psi(x,y)$ with $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Without invoking the no-slip condition, show that
(i) the streamlines are given by $\psi = \text{const}$,
(ii) $|\mathbf{u}| = |\nabla\psi|$, so that the flow is faster where the streamlines are closer,
(iii) the volume flux crossing any curve from $\mathbf{x}_0$ to $\mathbf{x}_1$ is given by $\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0)$,
(iv) $\psi = \text{const}$ on any *fixed* (i.e. stationary) boundary.
[Hint for (iii): $\mathbf{n}\,ds = (dy, -dx)$.]

8. Verify that the two-dimensional flow given in Cartesian coordinates by

$$u = \frac{y-b}{(x-a)^2 + (y-b)^2}, \qquad v = \frac{a-x}{(x-a)^2 + (y-b)^2}$$

satisfies $\nabla\cdot\mathbf{u} = 0$, and then find the streamfunction $\psi(x,y)$ such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Sketch the streamlines.

9. Verify that the two-dimensional flow given in polar coordinates by

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

satisfies $\nabla\cdot\mathbf{u} = 0$, and find the streamfunction $\psi(r,\theta)$. Sketch the streamlines, starting with $\psi = 0$.

$$\left[\text{Note:} \qquad \nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(u_\theta) \quad \text{and take} \quad u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{\partial\psi}{\partial r}\right]$$

10. A steady two-dimensional flow (pure straining) is given by $u = \alpha x$, $v = -\alpha y$ with $\alpha > 0$ and constant.
(i) Find the equation for a general streamline of the flow, and sketch some of them.
(ii) At $t = 0$ the fluid on the curve $x^2 + y^2 = a^2$ is marked (by an electro-chemical technique). Find the equation for this material fluid curve for $t > 0$.
(iii) Does the area within the curve change in time, and why?

11. Repeat question 10 for the two-dimensional flow (simple shear) given by $u = \gamma y$, $v = 0$ with $\gamma > 0$ and constant. Which of the two flows stretches the curve faster at long times?

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Example Sheet 2

Note to students and supervisors: Every answer should include a relevant sketch.

1. How high can water rise up one's arm hanging in the river from a lazy ($1\,\mathrm{m\,s^{-1}}$) punt? [*Hint: Use Bernoulli on surface streamline.*]

2. Waste water flows into an open-topped tank with volume flux Q and out of an exit pipe of small cross-sectional area A into the air. In steady state, how high above the pipe is the water in the tank?

3. A flat-bottomed barge moves very slowly through a closely fitting canal but generates a significant velocity U in the small gap beneath its bottom. Estimate how much lower the barge sits in the water compared to when it is stationary if $U = 5$ m s^{-1}.

4. Water from a large deep reservoir of depth D flows over a broad weir. Over the weir, the water is of depth $d(x) \ll D$ where the free surface has fallen to a level $h(x)$ below that far upstream in the reservoir, where x is downstream distance. Assume that the depth of water varies sufficiently slowly that the velocity is horizontal and uniform in depth and that dh/dx is non-zero at the crest of the weir. Show that the volume flux (per unit length normal to the flow) is $Q = d\sqrt{2gh}$. From the condition that Q does not vary along the flow, and the condition that $h+d$ is a minimum at the crest of the weir [differentiate], show that $h = \frac{1}{2}d$ at the crest. Deduce that $Q^2 = 8gL^3/27$ where L is the minimum value of $h + d$.

5. An axisymmetric jet of water of speed $1\,\mathrm{m\,s^{-1}}$ and cross-section $6 \times 10^{-4}\,\mathrm{m}^2$ strikes a wall at right angles and spreads out over it. By using the momentum integral equation over a suitable control volume and neglecting gravity, calculate the force on the wall due to the jet.

6. Starting from the Euler momentum equation for a fluid of constant density with a potential force $-\nabla\chi$, show that for a fixed volume V enclosed by surface A

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho u^2\,dV \;+\; \int_A H\mathbf{u}\cdot\mathbf{n}\,dA \;=\; 0$$

where $H = \frac{1}{2}\rho u^2 + p + \chi$ is the Bernoulli quantity, so concluding that H is the energy transported by the flow.

7. Using a Taylor expansion, show that, to leading order in small $\delta\mathbf{x}$, $\mathbf{u}(\mathbf{x}+\delta\mathbf{x}) - \mathbf{u}(\mathbf{x})$ can be written in suffix notation as $(E_{ij} + \frac{1}{2}\varepsilon_{jik}\omega_k)\delta x_j$ where $E_{ij} = E_{ji}$ and $\omega_k = (\nabla\times\mathbf{u})_k$. Find E_{ij} and ω_k for the case of linear shear flow $\mathbf{u} = (y, 0, 0)$ and sketch the streamlines of the flows $(\mathbf{u}_1)_i = E_{ij}x_j$ and $(\mathbf{u}_2)_i = \frac{1}{2}\varepsilon_{jik}\omega_k x_j$ for this case.

8. Calculate the vorticity of the velocity field

$$u = -\alpha x - yrf(t), \quad v = -\alpha y + xrf(t), \quad w = 2\alpha z$$

where $r^2 = x^2 + y^2$. Use the vorticity equation to deduce that $f(t) \propto \mathrm{e}^{3\alpha t}$. Explain the nature of this flow and describe the physical principle illustrated by your result.

9. If $\mathbf{u} = \mathbf{\Omega} \times \mathbf{x}$ (uniform rotation with angular velocity $\mathbf{\Omega}$) show that $\boldsymbol{\omega} = 2\mathbf{\Omega}$.

For a two-dimensional flow $\big(u(x, y), v(x, y), 0\big)$ show that $\boldsymbol{\omega} = (0, 0, -\nabla^2\psi)$, where ψ is the stream function.

A long cylinder filled with water has elliptical cross-section with major and minor semi-axes a and b. While $t < 0$ both the cylinder and the water within it rotate about the axis of the cylinder with uniform angular velocity $(0, 0, \Omega)$. What is the vorticity of the flow? Sketch the streamlines noting that they intersect the elliptical boundary of the cylinder. (Why?).

At $t = 0$ the cylinder is suddenly brought to rest. What is the vorticity for $t > 0$? Verify that the flow can be described by

$$\psi = \frac{a^2b^2\Omega}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

in suitable coordinates and sketch the streamlines.

10. A sphere of radius a moves with constant velocity U in a fluid otherwise at rest. How far ahead of the sphere is there a disturbance of magnitude $\frac{1}{20}U$? Show that the acceleration of a fluid particle at distance x ahead of the centre of the sphere is

$$3U^2\left(\frac{a^3}{x^4} - \frac{a^6}{x^7}\right).$$

11. Write down the velocity potential $\phi(x, y)$ for the two-dimensional flow produced by a point source of strength m located at the origin in a uniform stream $(U, 0)$. Show that there is a stagnation point at $(-a, 0)$, where $a = m/2\pi U$. Sketch the streamlines. Show that the streamfunction is given by $\psi = Uy + Ua\theta$, where θ is the polar angle from the positive x axis. From the sketch and the streamfunction show that ϕ represents the flow past a semi-infinite body whose width tends to $2\pi a$ far downstream.

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Example Sheet 3

Note to students and supervisors: Every answer should include a relevant sketch.

1. An orifice in the side of an open vessel containing water leads smoothly into a horizontal tube of uniform cross-section and length L. The diameter of the tube is small compared with L, with the horizontal dimensions of the free surface, and with the depth h of the orifice below the free surface. A plug at the end of the tube is suddenly removed and the water begins to flow. Show, using the expression for the pressure in unsteady irrotational flow, that the outflow velocity at subsequent times t is approximately

$$\sqrt{2gh}\tanh\left(\frac{t\sqrt{2gh}}{2L}\right).$$

Estimate the time scale for the flow in a garden hose to accelerate to its maximum velocity? (Assume that tap pressure is equivalent to ρgh with $h = 5$ m.)

2. A rigid disk of radius R is at a height $h(t)$ above a fixed horizontal plane $z = 0$, with incompressible, inviscid fluid filling the gap between them, and $h \ll R$. Given that the flow is axisymmetric and has a horizontal component that is independent of z, determine the velocity potential

$$\phi \;=\; \frac{\dot{h}}{4h}\left(2z^2 - r^2\right)$$

such that $\mathbf{u} = (u, 0, w) = \nabla\phi$ in cylindrical polar coordinates (r, θ, z). Ignoring gravity and assuming that the pressure at the edge of the disk is approximately constant, find the pressure distribution under the disk and hence the force on the fixed plane. Explain how the pressure distribution accelerates the radial flow in the cases $\dot{h} > 0$ and $\dot{h} < 0$.

3. A rigid sphere of radius a executes small-amplitude, linear oscillations (with a velocity $\mathbf{U}(t)$) about the centre of a larger fixed sphere of radius b. By approximating the boundary condition on the smaller sphere, neglecting terms quadratic in the amplitude, find the velocity potential for the induced motion of the fluid that fills the gap between the two spheres and show that the pressure distribution over the surface of the inner sphere is

$$\rho\dot{U}a\cos\theta(a^3 + \tfrac{1}{2}b^3)/(b^3 - a^3),$$

where θ is the angle with $\mathbf{U}$. Hence find the force exerted by the fluid on the inner sphere. Why is the force on the outer fixed sphere different? Comment on the case of a tight fit.

4. A U-tube consists of two long uniform vertical tubes of different cross-sectional areas A_1, A_2 connected at the base by a short tube of large cross-section, and contains an inviscid, incompressible fluid whose surface, in equilibrium, is at height h above the base. Derive the equation governing the nonlinear oscillations of the displacement ζ of the surface in the tube of cross-section A_2

$$(h + r\zeta)\frac{d^2\zeta}{dt^2} \;+\; \frac{r}{2}\left(\frac{d\zeta}{dt}\right)^2 \;+\; g\zeta \;=\; 0 \quad \text{where} \quad r = 1 - A_2/A_1.$$

5. Write down the equation and linearize the boundary conditions for the velocity potential and the motion of the free surface, for small amplitude oscillations of the water surface in a square container $0 \le x \le a$, $0 \le y \le a$ of depth h, i.e. water in $-h \le z \le \zeta(x,y,t)$ with $\zeta \ll 1$. Find the relationship between the frequency and the wave number of the oscillations. Show the sign of the surface displacement in plan view for the five lowest frequency modes.

6. Fluid of density ρ_1 occupies the region $z > 0$ and overlies another fluid of density ρ_2 (with $\rho_2 > \rho_1$), which occupies the region $z < 0$. Show that the frequencies of small amplitude oscillations of the interface between the regions are given by

$$\omega^2 = gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)$$

[*Hint: You will need different potentials ϕ_1 and ϕ_2 for the two regions and should apply the kinematic boundary condition to the flow in each region.*]

7. A rotating circular tank of radius a is filled with a volume V of fluid of density ρ. The tank is allowed to rotate for a long time, until the flow is a two-dimensional rigid-body motion with constant angular velocity Ω around the axis (so that $u = -\Omega y, v = \Omega x, w = 0$). Derive the equation for the pressure p at any point in the rotating fluid. What is the equation for the height $h(r)$ of the free surface?

[*Hint: Integrate the Euler equation to find the pressure and then use volume conservation to determine the constant of integration.*]

8. Wind blows steadily with uniform speed U from west to east over the United Kingdom. What is the magnitude and direction of the horizontal pressure gradient? Estimate the pressure difference between London and Edinburgh when $U = 10$ m s^{-1}.

[*You may need to look up values for the physical parameters.*]

9. Derive the linearized, rotating, shallow-water equations governing the horizontal flow (u, v) in a layer of inviscid fluid of depth $h_0 + \eta(x,y,t)$, where $\eta \ll h_0$, f is the Coriolis parameter and g is the acceleration due to gravity,

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x},$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y},$$

$$\frac{\partial \eta}{\partial t} + h_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

Consider the flow parallel to a coastline $x = 0$ with $u \equiv 0$ and show that

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0\frac{\partial^2 \eta}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 \eta}{\partial t \partial x} + f\frac{\partial \eta}{\partial y} = 0.$$

Find wavelike solutions of the form $\eta = A(x)B(y - ct)$ for arbitrary functions B, and determine the constant wave speed c. Given that $\eta \to 0$ as $x \to \infty$, determine the x-structure function A and explain why waves can only travel in the negative y direction.

[*These solutions represent coastally trapped Kelvin waves. They travel southwards along the east coast of England and northwards along the west coast of the Netherlands. One large such wave was responsible for the devastating floods in East Anglia and the Netherlands in 1953.*]

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