Part IB — Fluid Dynamics
Definitions

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Parallel viscous flow
Plane Couette flow, dynamic viscosity. Momentum equation and boundary conditions. Steady flows including Poiseuille flow in a channel. Unsteady flows, kinematic viscosity, brief description of viscous boundary layers (skin depth). [3]

Kinematics
Material time derivative. Conservation of mass and the kinematic boundary condition. Incompressibility; streamfunction for two-dimensional flow. Streamlines and path lines. [2]

Dynamics
Statement of Navier-Stokes momentum equation. Reynolds number. Stagnation-point flow; discussion of viscous boundary layer and pressure field. Conservation of momentum; Euler momentum equation. Bernoulli's equation.

Vorticity, vorticity equation, vortex line stretching, irrotational flow remains irrotational. [4]

Potential flows
Velocity potential; Laplace's equation, examples of solutions in spherical and cylindrical geometry by separation of variables. Translating sphere. Lift on a cylinder with circulation.

Expression for pressure in time-dependent potential flows with potential forces. Oscillations in a manometer and of a bubble. [3]

Geophysical flows
Linear water waves: dispersion relation, deep and shallow water, standing waves in a container, Rayleigh-Taylor instability.

Euler equations in a rotating frame. Steady geostrophic flow, pressure as streamfunction. Motion in a shallow layer, hydrostatic assumption, modified continuity equation. Conservation of potential vorticity, Rossby radius of deformation. [4]

Contents

0 Introduction

1 Parallel viscous flow

1.1 Preliminaries

Definition (Fluid)**.** A *fluid* is a material that flows.

Definition (Newtonian fluids and viscosity)**.** A *Newtonian fluid* is a fluid with a linear relationship between stress and rate of strain. The constant of proportionality is *viscosity.*

Definition (Stress)**.** *Stress* is force per unit area.

Definition (Strain)**.** *Strain* is the extension per unit length. The *rate of strain* is $\frac{\mathrm{d}}{\mathrm{d}t}(\text{strain})$ is concerned with gradients of velocity.

1.2 Stress

Definition (Normal stress)**.** The *normal stress* is

$$\tau_p = -p\mathbf{n}.$$

Definition (Tangential stress)**.** The *tangential stress* τ_s is the force (per unit area) required to move the top plate at speed U.

Definition (Dynamic viscosity)**.** The *dynamic viscosity* μ of the fluid is the constant of proportionality in

$$\tau_s = \mu \frac{U}{h}.$$

1.3 Steady parallel viscous flow

Definition (Steady flow)**.** A *steady flow* is a flow that does not change in time. In other words, all forces balance, and there is no acceleration.

Definition (Parallel flow)**.** A *parallel flow* is a flow where the fluid only flows in one dimension (say the x direction), and only depends on the direction perpendicular to a plane (say the $x - z$ plane). So the velocity can be written as

$$\mathbf{u} = (u(y), 0, 0).$$

1.4 Derived properties of a flow

Definition (Volume flux)**.** The *volume flux* is the volume of fluid traversing a cross-section per unit time. This is given by

$$q = \int_0^h u(y)\;\mathrm{d}y$$

per unit transverse width.

Definition (Vorticity)**.** The *vorticity* is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

1.5 More examples

Definition (Kinematic viscosity)**.** The *kinematic viscosity* is

$$\nu = \frac{\mu}{\rho}.$$

2 Kinematics

2.1 Material time derivative

Definition (Material derivative)**.** If $\mathbf{x}(t)$ is the (Lagrangian) path followed by a fluid particle, then necessarily $\dot{\mathbf{x}}(t) = \mathbf{u}$ by definition. In this case, we write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{D}f}{\mathrm{D}t}.$$

This is the *material derivative.*

In other words, we have

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \mathbf{u} \cdot \nabla f + \frac{\partial f}{\partial t}.$$

2.2 Conservation of mass

Definition (Incompressible fluid)**.** A fluid is *incompressible* if the density of a fluid particle does not change. This implies

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0,$$

and hence

$$\nabla \cdot \mathbf{u} = 0.$$

This is also known as the *continuity equation.*

2.3 Kinematic boundary conditions

2.4 Streamfunction for incompressible flow

Definition (Vector potential)**.** A *vector potential* is an $\mathbf{A}$ such that

$$\mathbf{u} = \nabla \times \mathbf{A}.$$

Definition (Streamfunction)**.** The ψ such that $\mathbf{A} = (0, 0, \psi)$ is the *streamfunction.*

Definition (Streamlines)**.** The *streamlines* are the contours of the streamfunction ψ.

3 Dynamics

3.1 Navier-Stokes equations

3.2 Pressure

3.3 Reynolds number

Definition (Reynolds number)**.** The *Reynolds number* is

$$Re = \frac{UL}{\nu},$$

which is a dimensionless number. This is a measure of the magnitude of the inertia to viscous terms.

Definition (Dynamic similarity)**.** Flows with the same geometry and equal Reynolds numbers are said to be dynamically similar.

3.4 A case study: stagnation point flow ($\mathbf{u} = \mathbf{0}$)

3.5 Momentum equation for inviscid ($\nu = 0$) incompressible fluid

3.6 Linear flows

3.7 Vorticity equation

4 Inviscid irrotational flow

Definition (Velocity potential)**.** The *velocity potential* of a velocity $\mathbf{u}$ is a scalar function ϕ such that $\mathbf{u} = \nabla\phi$.

Definition (Potential flow)**.** A *potential flow* is a flow whose velocity potential satisfies Laplace's equation.

4.1 Three-dimensional potential flows

4.2 Potential flow in two dimensions

4.3 Time dependent potential flows

5 Water waves

5.1 Dimensional analysis

5.2 Equation and boundary conditions

5.3 Two-dimensional waves (straight crested waves)

5.4 Group velocity

5.5 Rayleigh-Taylor instability

6 Fluid dynamics on a rotating frame

6.1 Equations of motion in a rotating frame

Definition (Coriolis parameter/planetary vorticity)**.** We conventionally write $2\mathbf{\Omega} = \mathbf{f}$, and we call this the *Coriolis parameter* or the *planetary vorticity.*

6.2 Shallow water equations

6.3 Geostrophic balance

Definition (Shallow water streamfunction)**.** The quantity

$$\psi = -\frac{gh}{f}$$

is the *shallow water streamfunction.*

Definition (Potential vorticity)**.** The *potential vorticity* is

$$\mathbf{Q} = \boldsymbol{\zeta} - \frac{\eta}{h_0}\mathbf{f},$$

Definition (Rossby radius of deformation)**.** The length scale

$$R = \frac{\sqrt{gh_0}}{f}$$

is the *Rossby radius of deformation.*