Part IB — Fluid Dynamics
Definitions

Based on lectures by P. F. Linden
Notes taken by Dexter Chua

Lent 2016

Parallel viscous flow
Plane Couette flow, dynamic viscosity. Momentum equation and boundary conditions. Steady flows including Poiseuille flow in a channel. Unsteady flows, kinematic viscosity, brief description of viscous boundary layers (skin depth). [3]

Kinematics
Material time derivative. Conservation of mass and the kinematic boundary condition. Incompressibility; streamfunction for two-dimensional flow. Streamlines and path lines. [2]

Dynamics
Statement of Navier-Stokes momentum equation. Reynolds number. Stagnation-point flow; discussion of viscous boundary layer and pressure field. Conservation of momentum; Euler momentum equation. Bernoulli's equation.

Vorticity, vorticity equation, vortex line stretching, irrotational flow remains irrotational. [4]

Potential flows
Velocity potential; Laplace's equation, examples of solutions in spherical and cylindrical geometry by separation of variables. Translating sphere. Lift on a cylinder with circulation.

Expression for pressure in time-dependent potential flows with potential forces. Oscillations in a manometer and of a bubble. [3]

Geophysical flows
Linear water waves: dispersion relation, deep and shallow water, standing waves in a container, Rayleigh-Taylor instability.

Euler equations in a rotating frame. Steady geostrophic flow, pressure as streamfunction. Motion in a shallow layer, hydrostatic assumption, modified continuity equation. Conservation of potential vorticity, Rossby radius of deformation. [4]

Contents

0 Introduction **3**

1 Parallel viscous flow **4**
1.1 Preliminaries 4
1.2 Stress 4
1.3 Steady parallel viscous flow 4
1.4 Derived properties of a flow 4
1.5 More examples 5

2 Kinematics **6**
2.1 Material time derivative 6
2.2 Conservation of mass 6
2.3 Kinematic boundary conditions 6
2.4 Streamfunction for incompressible flow 6

3 Dynamics **7**
3.1 Navier-Stokes equations 7
3.2 Pressure 7
3.3 Reynolds number 7
3.4 A case study: stagnation point flow ($\mathbf{u} = \mathbf{0}$) 7
3.5 Momentum equation for inviscid ($\nu = 0$) incompressible fluid . . 7
3.6 Linear flows 7
3.7 Vorticity equation 7

4 Inviscid irrotational flow **8**
4.1 Three-dimensional potential flows 8
4.2 Potential flow in two dimensions 8
4.3 Time dependent potential flows 8

5 Water waves **9**
5.1 Dimensional analysis 9
5.2 Equation and boundary conditions 9
5.3 Two-dimensional waves (straight crested waves) 9
5.4 Group velocity 9
5.5 Rayleigh-Taylor instability 9

6 Fluid dynamics on a rotating frame **10**
6.1 Equations of motion in a rotating frame 10
6.2 Shallow water equations 10
6.3 Geostrophic balance 10

0 Introduction

1 Parallel viscous flow

1.1 Preliminaries

Definition (Fluid)**.** A *fluid* is a material that flows.

Definition (Newtonian fluids and viscosity)**.** A *Newtonian fluid* is a fluid with a linear relationship between stress and rate of strain. The constant of proportionality is *viscosity.*

Definition (Stress)**.** *Stress* is force per unit area.

Definition (Strain)**.** *Strain* is the extension per unit length. The *rate of strain* is $\frac{\mathrm{d}}{\mathrm{d}t}$(strain) is concerned with gradients of velocity.

1.2 Stress

Definition (Normal stress)**.** The *normal stress* is

$$\tau_p = -p\mathbf{n}.$$

Definition (Tangential stress)**.** The *tangential stress* τ_s is the force (per unit area) required to move the top plate at speed U.

Definition (Dynamic viscosity)**.** The *dynamic viscosity* μ of the fluid is the constant of proportionality in

$$\tau_s = \mu \frac{U}{h}.$$

1.3 Steady parallel viscous flow

Definition (Steady flow)**.** A *steady flow* is a flow that does not change in time. In other words, all forces balance, and there is no acceleration.

Definition (Parallel flow)**.** A *parallel flow* is a flow where the fluid only flows in one dimension (say the x direction), and only depends on the direction perpendicular to a plane (say the $x - z$ plane). So the velocity can be written as

$$\mathbf{u} = (u(y), 0, 0).$$

1.4 Derived properties of a flow

Definition (Volume flux)**.** The *volume flux* is the volume of fluid traversing a cross-section per unit time. This is given by

$$q = \int_0^h u(y)\,\mathrm{d}y$$

per unit transverse width.

Definition (Vorticity)**.** The *vorticity* is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

1.5 More examples

Definition (Kinematic viscosity)**.** The *kinematic viscosity* is

$$\nu = \frac{\mu}{\rho}.$$

2 Kinematics

2.1 Material time derivative

Definition (Material derivative)**.** If $\mathbf{x}(t)$ is the (Lagrangian) path followed by a fluid particle, then necessarily $\dot{\mathbf{x}}(t) = \mathbf{u}$ by definition. In this case, we write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{D}f}{\mathrm{D}t}.$$

This is the *material derivative.*

In other words, we have

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \mathbf{u} \cdot \nabla f + \frac{\partial f}{\partial t}.$$

2.2 Conservation of mass

Definition (Incompressible fluid)**.** A fluid is *incompressible* if the density of a fluid particle does not change. This implies

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0,$$

and hence

$$\nabla \cdot \mathbf{u} = 0.$$

This is also known as the *continuity equation.*

2.3 Kinematic boundary conditions

2.4 Streamfunction for incompressible flow

Definition (Vector potential)**.** A *vector potential* is an $\mathbf{A}$ such that

$$\mathbf{u} = \nabla \times \mathbf{A}.$$

Definition (Streamfunction)**.** The ψ such that $\mathbf{A} = (0, 0, \psi)$ is the *streamfunction.*

Definition (Streamlines)**.** The *streamlines* are the contours of the streamfunction ψ.

3 Dynamics

3.1 Navier-Stokes equations

3.2 Pressure

3.3 Reynolds number

Definition (Reynolds number)**.** The *Reynolds number* is

$$Re = \frac{UL}{\nu},$$

which is a dimensionless number. This is a measure of the magnitude of the inertia to viscous terms.

Definition (Dynamic similarity)**.** Flows with the same geometry and equal Reynolds numbers are said to be dynamically similar.

3.4 A case study: stagnation point flow ($\mathbf{u} = \mathbf{0}$)

3.5 Momentum equation for inviscid ($\nu = 0$) incompressible fluid

3.6 Linear flows

3.7 Vorticity equation

4 Inviscid irrotational flow

Definition (Velocity potential)**.** The *velocity potential* of a velocity $\mathbf{u}$ is a scalar function ϕ such that $\mathbf{u} = \nabla\phi$.

Definition (Potential flow)**.** A *potential flow* is a flow whose velocity potential satisfies Laplace's equation.

4.1 Three-dimensional potential flows

4.2 Potential flow in two dimensions

4.3 Time dependent potential flows

5 Water waves

5.1 Dimensional analysis

5.2 Equation and boundary conditions

5.3 Two-dimensional waves (straight crested waves)

5.4 Group velocity

5.5 Rayleigh-Taylor instability

6 Fluid dynamics on a rotating frame

6.1 Equations of motion in a rotating frame

Definition (Coriolis parameter/planetary vorticity)**.** We conventionally write $2\mathbf{\Omega} = \mathbf{f}$, and we call this the *Coriolis parameter* or the *planetary vorticity*.

6.2 Shallow water equations

6.3 Geostrophic balance

Definition (Shallow water streamfunction)**.** The quantity

$$\psi = -\frac{gh}{f}$$

is the *shallow water streamfunction*.

Definition (Potential vorticity)**.** The *potential vorticity* is

$$\mathbf{Q} = \boldsymbol{\zeta} - \frac{\eta}{h_0}\mathbf{f},$$

Definition (Rossby radius of deformation)**.** The length scale

$$R = \frac{\sqrt{gh_0}}{f}$$

is the *Rossby radius of deformation*.