Part IB — Electromagnetism
Theorems with proof

Based on lectures by D. Tong
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Electromagnetism and Relativity
Review of Special Relativity; tensors and index notation. Lorentz force law. Electromagnetic tensor. Lorentz transformations of electric and magnetic fields. Currents and the conservation of charge. Maxwell equations in relativistic and non-relativistic forms. [5]

Electrostatics
Gauss's law. Application to spherically symmetric and cylindrically symmetric charge distributions. Point, line and surface charges. Electrostatic potentials; general charge distributions, dipoles. Electrostatic energy. Conductors. [3]

Magnetostatics
Magnetic fields due to steady currents. Ampre's law. Simple examples. Vector potentials and the Biot-Savart law for general current distributions. Magnetic dipoles. Lorentz force on current distributions and force between current-carrying wires. Ohm's law. [3]

Electrodynamics
Faraday's law of induction for fixed and moving circuits. Electromagnetic energy and Poynting vector. 4-vector potential, gauge transformations. Plane electromagnetic waves in vacuum, polarization. [5]

Contents

0 Introduction

1 Preliminaries

1.1 Charge and Current

Law (Continuity equation)**.**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

1.2 Forces and Fields

Law (Lorentz force law)**.**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Law (Maxwell's Equations)**.**

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{J}, \end{aligned}$$

where we have two constants of nature:

- $\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{m}^{-3}\,\mathrm{kg}^{-1}\,\mathrm{s}^2\,\mathrm{C}^2$ is the electric constant;
- $\mu_0 = 4\pi \times 10^{-6}\,\mathrm{m}\,\mathrm{kg}\,\mathrm{C}^{-2}$ is the magnetic constant.

Some prefer to call these constants the "permittivity of free space" and "permeability of free space" instead. But why bother with these complicated and easily-confused names when we can just call them "electric constant" and "magnetic constant"?

2 Electrostatics

2.1 Gauss' Law

Law (Gauss' law)**.**

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0},$$

where Q is the total charge inside V.

2.2 Electrostatic potential

2.2.1 Point charge

2.2.2 Dipole

2.2.3 General charge distribution

2.2.4 Field lines and equipotentials

2.3 Electrostatic energy

Proposition.

$$U = \frac{\varepsilon_0}{2} \int \mathbf{E} \cdot \mathbf{E} \, \mathrm{d}^3\mathbf{r}.$$

2.4 Conductors

3 Magnetostatics

3.1 Ampere's Law

Law (Ampere's law)**.**

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \mu_0 I,$$

where I is the current through the surface.

3.2 Vector potential

Proposition. We can always pick χ such that $\nabla \cdot \mathbf{A}' = 0$.

Proof. Suppose that $\mathbf{B} = \nabla \times \mathbf{A}$ with $\nabla \cdot \mathbf{A} = \psi(\mathbf{x})$. Then for any $\mathbf{A}' = \mathbf{A} + \nabla\chi$, we have

$$\nabla \cdot \mathbf{A}' = \nabla \mathbf{A} + \nabla^2\chi = \psi + \nabla^2\chi.$$

So we need a χ such that $\nabla^2\chi = -\psi$. This is the Poisson equation which we know that there is always a solution by, say, the Green's function. Hence we can find a χ that works. □

Law (Biot-Savart law)**.** The magnetic field is

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\,\mathrm{d}V'.$$

If the current is localized on a curve, this becomes

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\oint_C \mathrm{d}\mathbf{r}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3},$$

since $\mathbf{J}(\mathbf{r}')$ is non-zero only on the curve.

3.3 Magnetic dipoles

3.4 Magnetic forces

4 Electrodynamics

4.1 Induction

Law (Faraday's law of induction)**.**

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}.$$

4.2 Magnetostatic energy

Proposition. The energy stored in a magnetic field is

$$U = \frac{1}{2\mu_0}\int \mathbf{B}\cdot\mathbf{B}\,\mathrm{d}V.$$

4.3 Resistance

Law (Ohm's law)**.**

$$\mathcal{E} = IR,$$

Law (Ohm's law)**.**

$$\mathbf{J} = \sigma\mathbf{E}.$$

4.4 Displacement currents

4.5 Electromagnetic waves

4.6 Poynting vector

Theorem (Poynting theorem)**.**

$$\underbrace{\frac{\mathrm{d}U}{\mathrm{d}t} + \int_V \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V}_{\text{Total change of energy in } V \text{ (fields + particles)}} = \underbrace{-\frac{1}{\mu_0}\int_S (\mathbf{E}\times\mathbf{B})\cdot\mathrm{d}\mathbf{S}}_{\text{Energy that escapes through the surface } S}.$$

5 Electromagnetism and relativity

5.1 A review of special relativity

5.1.1 A geometric interlude on (co)vectors

5.1.2 Transformation rules

5.1.3 Vectors and covectors in SR

5.2 Conserved currents

5.3 Gauge potentials and electromagnetic fields

5.4 Maxwell Equations

5.5 The Lorentz force law