Electromagnetism: Example Sheet 1

Professor David Tong, January 2015

1. A current density $J(\mathbf{r},t)$ has the form

$$\mathbf{J} = C\,\mathbf{r}\,e^{-atr^2}$$

where C and a are constants. Show that the equation of conservation of charge can be satisfied by writing the charge density in the form

$$\rho \;=\; (f + t\,g)\; e^{-atr^2}$$

where f and g are functions of position, to be determined.

2. In a fluid environment, charge undergoes *diffusion*. This is empirically described by *Fick's law*, which relates the current to the charge density,

$$\mathbf{J} = -D\nabla\rho$$

Here, D is called the diffusion coefficient. Show that ρ obeys the heat equation. Show that this is solved by a spreading Gaussian of the form,

$$\rho(\mathbf{r},t) = \frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{3/2}}\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)$$

with a, t_0 and ρ_0 constants.

3. A charge density is given by $\rho = \rho_0\ e^{-k|z|}$ with ρ_0 and k positive constants. Use Gauss' law to show that the electric field is given by $\mathbf{E} = E(z)\hat{\mathbf{z}}$ with $E(z) = -E(-z)$ and, for $z > 0$,

$$E(z) = \frac{\rho_0}{\epsilon_0 k}\left(1 - e^{-kz}\right)$$

4. Use Gauss's law to obtain the electric field due to a uniform charge density ρ occupying the region $a < r < b$, with r the radial distance from the origin.

Show that in the limit $b \to a$, $\rho \to \infty$ with $(b-a)\,\rho = \sigma$ remaining finite, the electric field suffers the expected discontinuity due to surface charge.

5. Roughly sketch the field lines (including arrows to denote sense) and equipotentials for the following systems of point charges:

- A single charge $+q$;
- Two charges $+q$ separated by a distance $2a$;
- Two charges $\pm q$ separated by a distance $2a$.

6. Compute the electric field due to an infinite line charge by integrating the expression obtained from the inverse square law.

7. A circular disk of radius a has uniform surface charge density σ. Compute the potential at a point on the axis of symmetry at distance z from the centre. Compute the electric field at this point. Find the discontinuity in the normal electric field at the centre of the disk. Show that, far along the axis of symmetry, the electric field looks approximately like that of a charged point particle.

8. Show that, far from a charge distribution $\rho(\mathbf{r})$ localised in region V, the potential takes the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} + \frac{1}{2}\frac{\mathbb{Q}_{ij}r_ir_j}{r^5} + \dots\right)$$

where Q is the total charge, $\mathbf{p}$ is the dipole moment and $\mathbb{Q}_{ij}$ is the quadrupole moment, defined by

$$Q = \int_V d^3r\ \rho(\mathbf{r}) \quad \text{and} \quad \mathbf{p} = \int_V d^3r\ \mathbf{r}\rho(\mathbf{r}) \quad \text{and} \quad \mathbb{Q}_{ij} = \int_V d^3r\ (3r_ir_j - \delta_{ij}r^2)\rho(\mathbf{r})$$

Compute the charge, dipole and quadrupole for:

- Two charges, $+q$ and $-q$, at points $(0,0,0)$ and $(d,0,0)$ respectively.
- Two charges $+q$ and two charges $-q$ placed on the corners of a square, with sides of length d, such that every charge has an opposite charge for each of its neighbours.
- Four charges $+q$ and four charges $-q$ placed on the corners of a cube, with sides of length d, such that every charge has an opposite charge for each of its neighbours.

9. For a charge density $\rho(\mathbf{x}, t)$ and current $\mathbf{J}(\mathbf{x}, t)$, both localised within a region V, show that

$$\int_V d^3x \; \mathbf{J} = \frac{d\mathbf{p}}{dt}$$

where $\mathbf{p}$ is the electric dipole moment.

[*Hint*: You may wish to first show that $\partial_j(x_i J_j) = J_i + x_i \nabla \cdot \mathbf{J}$]

10. A charge density $\rho(\mathbf{r})$ gives rise to a potential $\phi(\mathbf{r})$. In the lectures we derived the expression for the energy stored in the field,

$$U = \frac{1}{2} \int_{\mathbf{R}^3} d^3r \; \rho\phi$$

Show that this is equivalent to the formula

$$U = \frac{\epsilon_0}{2} \int_{\mathbf{R}^3} d^3r \; \mathbf{E} \cdot \mathbf{E}$$

Consider a total charge Q, distributed uniformly inside a sphere of radius R. Find ϕ and $\mathbf{E}$ both inside and outside the sphere. Calculate the two expressions for the energy and show that they agree.

11*. A spherical conducting shell of radius R is grounded (i.e. has potential $\phi = 0$). A charge q is placed inside the shell at at point $\mathbf{r} = (0, 0, d)$ from the centre, with $d < R$. Show that the potential inside the shell can be determined by placing an appropriate image charge outside the shell at $\mathbf{r}' = (0, 0, R^2/d)$. Show that the induced surface charge on the conductor is

$$\sigma = -\frac{q}{4\pi} \frac{R^2 - d^2}{R(R^2 - 2dR\cos\theta + d^2)^{3/2}}$$

where θ is the angle between the point on the shell and the z-axis.

Electromagnetism: Example Sheet 2

Professor David Tong, February 2015

1. A constant magnetic field points along the z-axis: $\mathbf{B} = B\hat{\mathbf{z}}$. Verify that each of the following vector potentials satisfies $\mathbf{B} = \nabla \times \mathbf{A}$:

- $\mathbf{A} = xB\hat{\mathbf{y}}$
- $\mathbf{A} = \frac{1}{2}(xB\hat{\mathbf{y}} - yB\hat{\mathbf{x}})$
- In cylindrical polar coordinates, $\mathbf{A} = \frac{1}{2}rB\hat{\boldsymbol{\phi}}$, with $r^2 = x^2 + y^2$
- In spherical polar coordinates, $\mathbf{A} = \frac{1}{2}r\sin\theta\, B\hat{\boldsymbol{\phi}}$, with $r^2 = x^2 + y^2 + z^2$.

2. A cylindrical conductor of radius a, with axis along the z-axis, carries a uniform current density $\mathbf{J} = J\hat{\mathbf{z}}$. Use Ampère's law to show that the magnetic field within the conductor is given, in cylindrical polar coordinates, by

$$\mathbf{B} = \frac{1}{2}\mu_0 J r\hat{\boldsymbol{\phi}}$$

with $r^2 = x^2 + y^2$. [In this question, and the following question, you may assume that the magnetic field inside a conductor is the same as in a vacuum.]

3. A steady current I flows in the z-direction uniformly in the the region between the cylinders $x^2 + y^2 = a^2$ and $(x+d)^2 + y^2 = b^2$, where $0 < d < (b-a)$. Show that the associated magnetic field $\mathbf{B}$ throughout the region $x^2 + y^2 < a^2$ is given by

$$\mathbf{B} = \frac{\mu_0 I d}{2\pi(b^2 - a^2)}\hat{\mathbf{y}}$$

4. Use the Biot-Savart law to determine the magnetic field:

- Around an infinite, straight wire carrying current I.
- At the centre of a square loop of wire, with sides of length a, carrying current I.
- At the point $(0, 0, z)$ above a loop of wire of radius a, lying in the (x, y) plane, with centre at the origin, carrying current I.

5. Explain why the force $\mathbf{F}$ and torque $\boldsymbol{\tau}$ experienced by a loop of wire C carrying current I are given by

$$\mathbf{F} = I \oint_C d\mathbf{r} \times \mathbf{B} \quad \text{and} \quad \boldsymbol{\tau} = I \oint_C \mathbf{r} \times (d\mathbf{r} \times \mathbf{B})$$

A loop of wire lies in a plane whose normal makes an angle θ with a uniform magnetic field. The loop of wire encloses a planar area A and carries current I. Compute the torque.

6. What boundary conditions apply on either side of a surface current $\mathbf{K}$?

A surface current experiences a Lorentz force from the *average* magnetic field on either side of the surface. A wire carrying current I winds N times per unit length to form a cylindrical solenoid. Show that there is a force per unit area on the cylinder given by

$$\mathbf{f} = \frac{\mu_0 I^2 N^2}{2} \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the outward normal.

7⋆. A steady current I_1 flows around a closed loop C_1. Use the Biot-Savart law to show that this exerts a force on a second loop C_2 carrying current I_2, given by

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_2} \oint_{C_1} d\mathbf{r}_2 \times \left(d\mathbf{r}_1 \times \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \right)$$

Write this in a form which exhibits anti-symmetry, $\mathbf{F}_{12} = -\mathbf{F}_{21}$, in agreement with Newton's third law.

8. A current creates a time-dependent electric and magnetic field given, in cylindrical polar coordinates, by

$$\mathbf{E} = e^{-t} \hat{\boldsymbol{\phi}} \quad , \quad \mathbf{B} = \frac{e^{-t}}{r} \hat{\mathbf{z}}$$

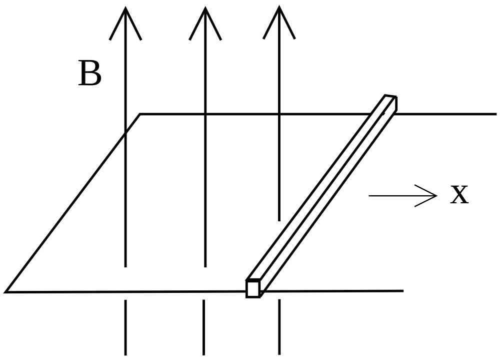
(Here $r^2 = x^2 + y^2$). Verify that these are consistent with the remaining Maxwell equations $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial \mathbf{B}/\partial t = 0$.

The emf around a moving circuit $C(t)$ is given by

$$\oint_{C(t)} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = -\frac{d}{dt} \int_{S(t)} \mathbf{B} \cdot d\mathbf{S}$$

where $S(t)$ is the surface spanning $C(t)$ and $\mathbf{v}$ is the velocity of a point on the circuit. Verify this equation by explicitly evaluating the integrals for a circle $C(t)$ lying in the plane $z = 0$ with radius $R(t) = 1 + t$ in the electric and magnetic fields given above.

9. A horizontal, rectangular circuit, shown in the figure, has a sliding bar of mass m and length L which moves, without friction, in the x-direction. The bar and all the wires in the circuit have resistance R per unit length.



A uniform vertical magnetic field $\mathbf{B} = (\alpha/t)\hat{\mathbf{z}}$ is applied for time $t > 0$, with α constant. Derive the differential equation satisfied by the position x for $t > 0$. Find a solution.

[In this question, and the following question, you may assume that the effect on the magnetic field due to any current flow is negligible compared to the background $\mathbf{B}$.]

10. A vector potential is given, in cylindrical polar coordinates, by $A_\phi = \frac{1}{2}Brz$ where B is constant (and, again, $r^2 = x^2 + y^2$). Compute the magnetic field $\mathbf{B}$.

A conducting loop of radius a and resistance R lies in the (x, y) plane at position $z(t)$, its centre on the axis. Find the induced current in the loop.

Compute the force exerted on the loop by the magnetic field. To overcome this, an equal and opposite force is applied to the loop. Show that the work done per unit time by this force is equal to the rate of dissipation of energy due to the resistance in the loop.

Electromagnetism: Example Sheet 3

Professor David Tong, February 2015

1. A steady current I flows along a cylindrical conductor of constant circular cross-section and uniform conductivity σ. Show, using the relevant equations for $\mathbf{E}$ and $\mathbf{J}$, that the current is distributed uniformly across the cross-section of the cylinder, and calculate the electric and magnetic fields just outside the surface of the cylinder.

Verify that the integral of the Poynting vector over unit length of the surface is equal to the rate per unit length of dissipation of electrical energy as heat.

2. A monochromatic wave with fields

$$\mathbf{E}_{\rm inc} = E_0 \hat{\mathbf{x}}\, e^{i(kz-\omega t)} \quad , \quad \mathbf{B}_{\rm inc} = \frac{E_0}{c}\hat{\mathbf{y}}\, e^{i(kz-\omega t)}$$

propagates in empty space $z < 0$. A perfect conductor fills the region $z \geq 0$. Show that if the reflected fields are given by

$$\mathbf{E}_{\rm ref} = -E_0 \hat{\mathbf{x}}\, e^{i(-kz-\omega t)} \quad , \quad \mathbf{B}_{\rm ref} = \frac{E_0}{c}\hat{\mathbf{y}}\, e^{i(-kz-\omega t)}$$

then the total fields $\mathbf{E} = \mathbf{E}_{\rm inc} + \mathbf{E}_{\rm ref}$ and $\mathbf{B} = \mathbf{B}_{\rm inc} + \mathbf{B}_{\rm ref}$ satisfy the Maxwell equations and the relevant boundary conditions at $z = 0$.

What surface current flows in the plane $z = 0$? Compute the Poynting vector in the region $z < 0$ and compute its value averaged over a period $T = 2\pi/\omega$.

Recall from Q6, Sheet 2, that a surface current experiences a Lorentz force from the average magnetic field on either side of the surface. Use this to show that the time-averaged force per unit area on the conductor is $\bar{f} = \epsilon_0 E_0^2$.

3. Perfectly conducting planes are positioned at $y = 0$ and $y = a$. Show that a monochromatic plane wave can propagate between the plates in the y direction only if the frequency is given by $\omega = n\pi c/a$ with $n \in \mathbf{Z}$.

4. Perfectly conducting planes are positioned at $y = 0$ and $y = a$. Show that a monochromatic wave may propagate between the plates in the direction z if the field components are

$$\begin{aligned} E_x &= \omega A \sin\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t) \\ B_y &= kA \sin\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t) \\ B_z &= \frac{n\pi A}{a}\cos\left(\frac{n\pi y}{a}\right)\cos(kz-\omega t) \end{aligned}$$

with A a constant and $n \in \mathbf{Z}$. Show that the wavelength λ is given by

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_\infty^2} - \frac{n^2}{4a^2}$$

where λ_∞ is the wavelength of waves of the same frequency in the absence of conducting plates.

5. Consider a plane polarized electromagnetic wave described by the vector and scalar potentials,

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad \text{and} \quad \phi(\mathbf{r},t) = \phi_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

with constant $\mathbf{A}_0$ and ϕ_0. Use Maxwell's equations to find a relationship between $\mathbf{A}_0$ and ϕ_0.

Find a gauge transformation such that the new vector potential is "transversely polarised", i.e. $\mathbf{A}_0 \cdot \mathbf{k} = 0$. What is the scalar potential ϕ in this gauge?

6. For constant electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, show that if $\mathbf{E}\cdot\mathbf{B} = 0$ and $\mathbf{E}^2 - c^2\mathbf{B}^2 \neq 0$ then there exist frames of reference where either $\mathbf{E}$ or $\mathbf{B}$ are zero, but not both.

[Hint: it suffices to take just E_y and B_z non zero and consider Lorentz transformations along the x-direction with speed $v < c$.]

7. An electromagnetic wave is reflected by a perfect conductor at $x = 0$. The electric field has the form

$$\mathbf{E}(t,\mathbf{x}) = \widehat{\mathbf{y}}\,[f(t_-) - f(t_+)]$$

where f is an arbitrary function and $ct_\pm = ct \pm x$. Show that this satisfies the relevant boundary condition at the conductor. Find the corresponding magnetic field $\mathbf{B}$.

Show that under a Lorentz transformation to a frame moving with speed v in the x-direction the electric field is transformed to

$$\mathbf{E}'(t',\mathbf{x}') = \widehat{\mathbf{y}}\left[\rho f(\rho t'_-) - \frac{1}{\rho} f\left(\frac{t'_+}{\rho}\right)\right] \quad \text{where} \quad \rho = \sqrt{\frac{c-v}{c+v}}$$

Hence for an incident wave $\mathbf{E}(t,\mathbf{x}) = \widehat{\mathbf{y}}F(t_-)$, find the wave that is reflected after it hits a perfectly conducting mirror moving with speed v in the x-direction.

8. In $d+1$ space-time dimensions, the equations of electromagnetism are given by

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{where} \quad \mu,\nu = 0,1,\ldots d$$

How many components does the electric field have? How many components does the magnetic field have? What is the potential energy between two electric charges q_1 and q_2? How many independent, linear polarisations does an electromagnetic wave have?

[Note: Pay particular attention to the cases $d=1$ and $d=2$, partly because they're special and partly because they can actually be realised in experiment. For $d \geq 4$, you may denote the area of a $(d-1)$-dimensional sphere as S_{d-1}.]

9. A particle of rest mass m and charge q moves in a constant uniform electric field $\mathbf{E} = (E,0,0)$. It starts from the origin with initial momentum $\mathbf{p} = (0,p_0,0)$. Show that the particle traces out a path in the (x,y) plane given by

$$x = \frac{\mathcal{E}_0}{qE}\left(\cosh\left(\frac{qEy}{p_0 c}\right) - 1\right)$$

where $\mathcal{E}_0 = \sqrt{p_0^2 c^2 + m^2 c^4}$ is the initial kinematic energy of the particle.

10*. For a general 4-velocity, written as $U_\mu = \gamma(c, -\mathbf{v})$, show that

$$F^{\mu\nu}U_\nu = \gamma \begin{pmatrix} \mathbf{E}\cdot\mathbf{v}/c \\ \mathbf{E} + \mathbf{v}\times\mathbf{B} \end{pmatrix}$$

In the rest-frame of a conducting medium, Ohm's law states that $\mathbf{J} = \sigma\mathbf{E}$ where σ is the conductivity and $\mathbf{J}$ is the 3-current. Assuming that σ is a Lorentz scalar, show that Ohm's law can be written covariantly as

$$J^\mu - \frac{1}{c^2}(J^\nu U_\nu)U^\mu = \sigma F^{\mu\nu}U_\nu$$

where J^μ is the 4-current and U^μ is the (uniform) 4-velocity of the medium. If the medium moves with 3-velocity $\mathbf{v}$ in some inertial frame, show that the current in that frame is

$$\mathbf{J} = \rho\mathbf{v} + \sigma\gamma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{1}{c^2}(\mathbf{v}\cdot\mathbf{E})\mathbf{v}\right)$$

where ρ is the charge density. Simplify this formula, given that the charge density vanishes in the rest-frame of the medium.