Part IB — Complex Methods
Theorems with proof

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Analytic functions
Definition of an analytic function. Cauchy-Riemann equations. Analytic functions as conformal mappings; examples. Application to the solutions of Laplace's equation in various domains. Discussion of $\log z$ and z^a. [5]

Contour integration and Cauchy's Theorem
[*Proofs of theorems in this section will not be examined in this course.*]
Contours, contour integrals. Cauchy's theorem and Cauchy's integral formula. Taylor and Laurent series. Zeros, poles and essential singularities. [3]

Residue calculus
Residue theorem, calculus of residues. Jordan's lemma. Evaluation of definite integrals by contour integration. [4]

Fourier and Laplace transforms
Laplace transform: definition and basic properties; inversion theorem (proof not required); convolution theorem. Examples of inversion of Fourier and Laplace transforms by contour integration. Applications to differential equations. [4]

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0 Introduction

1 Analytic functions

1.1 The complex plane and the Riemann sphere

1.2 Complex differentiation

Proposition (Cauchy-Riemann equations)**.** If $f = u + iv$ is differentiable, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proposition. Given a complex function $f = u + iv$, if u and v are real differentiable at a point z and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then f is differentiable at z.

1.3 Harmonic functions

Proposition. The real and imaginary parts of any analytic function are harmonic.

1.4 Multi-valued functions

1.5 Möbius map

Proposition. Möbius maps take circlines to circlines.

Proof. Any circline can be expressed as a circle of Apollonius,

$$|z - z_1| = \lambda |z - z_2|,$$

where $z_1, z_2 \in \mathbb{C}$ and $\lambda \in \mathbb{R}^+$.

This was proved in the first example sheet of IA Vectors and Matrices. The case $\lambda = 1$ corresponds to a line, while $\lambda \neq 1$ corresponds to a circle. Substituting z in terms of w, we get

$$\left|\frac{-dw + b}{cw - a} - z_1\right| = \lambda \left|\frac{-dw + b}{cw - a} - z_2\right|.$$

Rearranging this gives

$$|(cz_1 + d)w - (az_1 + b)| = \lambda |(cz_2 + d)w - (az_2 + b)|. \qquad (*)$$

A bit more rearranging gives

$$\left|w - \frac{az_1 + b}{cz_1 + d}\right| = \lambda \left|\frac{cz_2 + d}{cz_1 + d}\right| \left|w - \frac{az_2 + b}{cz_2 + d}\right|.$$

This is another circle of Apollonius.

Note that the proof fails if either $cz_1 + d = 0$ or $cz_2 + d = 0$, but then $(*)$ trivially represents a circle. □

Proposition. Given six points $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{C}^*$, we can find a Möbius map which sends $\alpha \mapsto \alpha', \beta \mapsto \beta'$ and $\gamma \to \gamma'$.

Proof. Define the Möbius map

$$f_1(z) = \frac{\beta - \gamma}{\beta - \alpha}\frac{z - \alpha}{z - \gamma}.$$

By direct inspection, this sends $\alpha \to 0, \beta \to 1$ and $\gamma \to \infty$. Again, we let

$$f_2(z) = \frac{\beta' - \gamma'}{\beta' - \alpha'}\frac{z - \alpha'}{z - \gamma'}.$$

This clearly sends $\alpha' \to 0, \beta' \to 1$ and $\gamma' \to \infty$. Then $f_2^{-1} \circ f_1$ is the required mapping. It is a Möbius map since Möbius maps form a group. □

1.6 Conformal maps

Proposition. A conformal map preserves the angles between intersecting curves.

Proof. Suppose $z_1(t)$ is a curve in $\mathbb{C}$, parameterised by $t \in \mathbb{R}$, which passes through a point z_0 when $t = t_1$. Suppose that its tangent there, $z_1'(t_1)$, has a well-defined direction, i.e. is non-zero, and the curve makes an angle $\phi = \arg z_1'(t_1)$ to the x-axis at z_0.

Consider the image of the curve, $Z_1(t) = f(z_1(t))$. Its tangent direction at $t = t_1$ is

$$Z_1'(t_1) = z_1'(t_1)f'(z_1(t_1)) = z_1'(t_0)f'(z_0),$$

and therefore makes an angle with the x-axis of

$$\arg(Z_1'(t_1)) = \arg(z_1'(t_1)f'(z_0)) = \phi + \arg f'(z_0),$$

noting that $\arg f'(z_0)$ exists since f is conformal, and hence $f'(z_0) \neq 0$.

In other words, the tangent direction has been rotated by $\arg f'(z_0)$, and this is independent of the curve we started with.

Now if $z_2(t)$ is another curve passing through z_0. Then its tangent direction will also be rotated by $\arg f'(z_0)$. The result then follows. □

1.7 Solving Laplace's equation using conformal maps

2 Contour integration and Cauchy's theorem

2.1 Contour and integrals

Proposition.

(i) We write $\gamma_1 + \gamma_2$ for the path obtained by joining γ_1 and γ_2. We have

$$\int_{\gamma_1+\gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z$$

Compare this with the equivalent result on the real line:

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x.$$

(ii) Recall $-\gamma$ is the path obtained from reversing γ. Then we have

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z.$$

Compare this with the real result

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$$

(iii) If γ is a contour from a to b in $\mathbb{C}$, then

$$\int_\gamma f'(z)\,\mathrm{d}z = f(b) - f(a).$$

This looks innocuous. This is just the fundamental theorem of calculus. However, there is some subtlety. This requires f to be differentiable at every point on γ. In particular, it must not cross a branch cut. For example, our previous example had $\log z$ as the antiderivative of $\frac{1}{z}$. However, this does not imply the integrals along different paths are the same, since we need to pick different branches of log for different paths, and things become messy.

(iv) Integration by substitution and by parts work exactly as for integrals on the real line.

(v) If γ has length L and $|f(z)|$ is bounded by M on γ, then

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq LM.$$

This is since

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \int_\gamma |f(z)||\,\mathrm{d}z| \leq M\int_\gamma |\mathrm{d}z| = ML.$$

We will be using this result a lot later on.

2.2 Cauchy's theorem

Theorem (Cauchy's theorem)**.** If $f(z)$ is analytic in a simply-connected domain $\mathcal{D}$, then for every simple closed contour γ in $\mathcal{D}$, we have

$$\oint_\gamma f(z)\ \mathrm{d}z = 0.$$

Proof. (non-examinable) The proof of this remarkable theorem is simple (with a catch), and follows from the Cauchy-Riemann equations and Green's theorem. Recall that Green's theorem says

$$\oint_{\partial S}(P\ \mathrm{d}x + Q\ \mathrm{d}y) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ \mathrm{d}x\ \mathrm{d}y.$$

Let u, v be the real and imaginary parts of f. Then

$$\begin{aligned}\oint_\gamma f(z)\ \mathrm{d}z &= \oint_\gamma (u + iv)(\mathrm{d}x + i\ \mathrm{d}y)\\ &= \oint_\gamma (u\ \mathrm{d}x - v\ \mathrm{d}y) + i\oint_\gamma (v\ \mathrm{d}x + u\ \mathrm{d}y)\\ &= \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\ \mathrm{d}x\ \mathrm{d}y + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\ \mathrm{d}x\ \mathrm{d}y\end{aligned}$$

But both integrands vanish by the Cauchy-Riemann equations, since f is differentiable throughout S. So the result follows. □

2.3 Contour deformation

Proposition. Suppose that γ_1 and γ_2 are contours from a to b, and that f is analytic on the contours *and* between the contours. Then

$$\int_{\gamma_1} f(z)\ \mathrm{d}z = \int_{\gamma_2} f(z)\ \mathrm{d}z.$$

Proof. Suppose first that γ_1 and γ_2 do not cross. Then $\gamma_1 - \gamma_2$ is a simple closed contour. So

$$\oint_{\gamma_1 - \gamma_2} f(z)\ \mathrm{d}z = 0$$

by Cauchy's theorem. Then the result follows.

If γ_1 and γ_2 *do* cross, then dissect them at each crossing point, and apply the previous result to each section. □

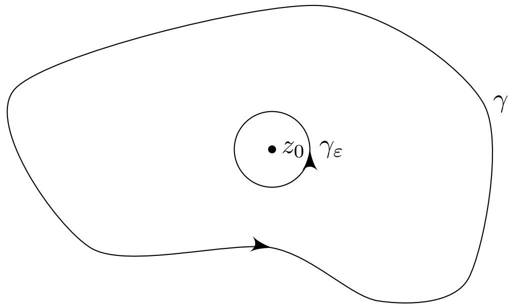
2.4 Cauchy's integral formula

Theorem (Cauchy's integral formula)**.** Suppose that $f(z)$ is analytic in a domain $\mathcal{D}$ and that $z_0 \in \mathcal{D}$. Then

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\ \mathrm{d}z$$

for any simple closed contour γ in $\mathcal{D}$ encircling z_0 anticlockwise.

Proof. (non-examinable) We let γ_ε be a circle of radius ε about z_0, within γ.



Since $\frac{f(z)}{z-z_0}$ is analytic except when $z = z_0$, we know

$$\oint_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z = \oint_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\,\mathrm{d}z.$$

We now evaluate the right integral directly. Substituting $z = z_0 + \varepsilon e^{i\theta}$, we get

$$\begin{aligned}\oint_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\,\mathrm{d}z &= \int_0^{2\pi} \frac{f(z_0+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta}\,\mathrm{d}\theta\\ &= i\int_0^2 \pi(f(z_0) + O(\varepsilon))\,\mathrm{d}\theta\\ &\to 2\pi i f(z_0)\end{aligned}$$

as we take the limit $\varepsilon \to 0$. The result then follows. □

Theorem (Liouville's theorem*)**.** Any bounded entire function is a constant.

Proof. (non-examinable) Suppose that $|f(z)| \leq M$ for all z, and consider a circle of radius r centered at an arbitrary point $z_0 \in \mathbb{C}$. Then

$$f'(z_0) = \frac{1}{2\pi i}\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z.$$

Hence we know

$$\frac{1}{2\pi i} \leq \frac{1}{2\pi}\cdot 2\pi r \cdot \frac{M}{r^2} \to 0$$

as $r \to \infty$. So $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. So f is constant. □

3 Laurent series and singularities

3.1 Taylor and Laurent series

Proposition (Laurent series)**.** If f is analytic in an *annulus* $R_1 < |z - z_0| < R_2$, then it has a *Laurent series*

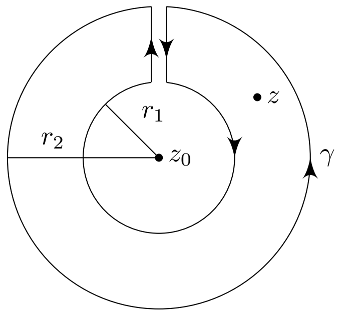
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

This is convergent within the annulus. Moreover, the convergence is uniform within compact subsets of the annulus.

Proof. (non-examinable) We wlog $z_0 = 0$. Given a z in the annulus, we pick r_1, r_2 such that

$$R_1 < r_1 < |z| < r_2 < R_2,$$

and we let γ_1 and γ_2 be the contours $|z| = r_1$, $|z| = r_2$ traversed anticlockwise respectively. We choose γ to be the contour shown in the diagram below.



We now apply Cauchy's integral formula (after a change of notation):

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

We let the distance between the cross-cuts tend to zero. Then we get

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}z - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta,$$

We have to subtract the second integral because it is traversed in the opposite direction. We do the integrals one by one. We have

$$\oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z} = -\frac{1}{z} \oint_{\gamma_1} \frac{f(\zeta)}{1 - \frac{\zeta}{z}}\,\mathrm{d}\zeta$$

Taking the Taylor series of $\frac{1}{1 - \frac{\zeta}{z}}$, which is valid since $|\zeta| = r_1 < |z|$ on γ_1, we obtain

$$\begin{aligned} &= -\frac{1}{z} \oint_{\gamma_1} f(\zeta) \sum_{m=0}^{\infty} \left(\frac{\zeta}{z}\right)^m \mathrm{d}\zeta \\ &= -\sum_{m=0}^{\infty} z^{-m-1} \oint_{\gamma_1} f(\zeta)\zeta^m\,\mathrm{d}\zeta. \end{aligned}$$

This is valid since $\oint \sum = \sum \oint$ by uniform convergence. So we can deduce

$$-\frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \sum_{n=-\infty}^{-1} a_n z^n,$$

where

$$a_n = \frac{1}{2\pi i}\oint_{\gamma_1} f(\zeta)\zeta^{-n-1}\,\mathrm{d}\zeta$$

for $n < 0$.

Similarly, we obtain

$$\frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \sum_{n=0}^{\infty} a_n z^n,$$

for the same definition of a_n, except $n \geq 0$, by expanding

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\frac{1}{1 - \frac{z}{\zeta}} = \sum_{n=0}^{\infty}\frac{z^n}{\zeta^{n+1}}.$$

This is again valid since $|\zeta| = r_2 > |z|$ on γ_2. Putting these result together, we obtain the Laurent series. The general result then follows by translating the origin by z_0.

We will not prove uniform convergence — go to IB Complex Analysis. □

3.2 Zeros

3.3 Classification of singularities

3.4 Residues

Proposition. At a *simple* pole, the residue is given by

$$\operatorname*{res}_{z=z_0} f(z) = \lim_{z\to z_0}(z - z_0)f(z).$$

Proof. We can simply expand the right hand side to obtain

$$\begin{aligned}\lim_{z\to z_0}(z - z_0)\left(\frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots\right) &= \lim_{z\to z_0}(a_{-1} + a_0(z - z_0) + \cdots)\\ &= a_{-1},\end{aligned}$$

as required. □

Proposition. At a pole of order N, the residue is given by

$$\lim_{z\to z_0}\frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}(z - z_0)^N f(z).$$

Theorem. γ be an anticlockwise simple closed contour, and let f be analytic within γ except for an isolated singularity z_0. Then

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i a_{-1} = 2\pi i \operatorname*{res}_{z=z_0} f(z).$$

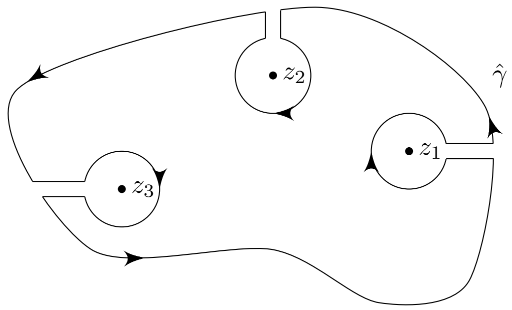
4 The calculus of residues

4.1 The residue theorem

Theorem (Residue theorem)**.** Suppose f is analytic in a simply-connected region except at a finite number of isolated singularities $z_1, \cdots, z_n$, and that a simple closed contour γ encircles the singularities anticlockwise. Then

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^n \operatorname*{res}_{z=z_k} f(z).$$

Proof. Consider the following curve $\hat{\gamma}$, consisting of small clockwise circles $\gamma_1, \cdots, \gamma_n$ around each singularity; cross cuts, which cancel in the limit as they approach each other, in pairs; and the large outer curve (which is the same as γ in the limit).



Note that $\hat{\gamma}$ encircles no singularities. So $\oint_{\hat{\gamma}} f(z)\,\mathrm{d}z = 0$ by Cauchy's theorem. So in the limit when the cross cuts cancel, we have

$$\oint_\gamma f(z)\,\mathrm{d}z + \sum_{k=1}^n \oint_{\gamma_k} f(z)\,\mathrm{d}z = \oint_{\hat{\gamma}} f(z)\,\mathrm{d}z = 0.$$

But from what we did in the previous section, we know

$$\oint_{\gamma_k} f(z)\,\mathrm{d}z = -2\pi i \operatorname*{res}_{z=z_k} f(z),$$

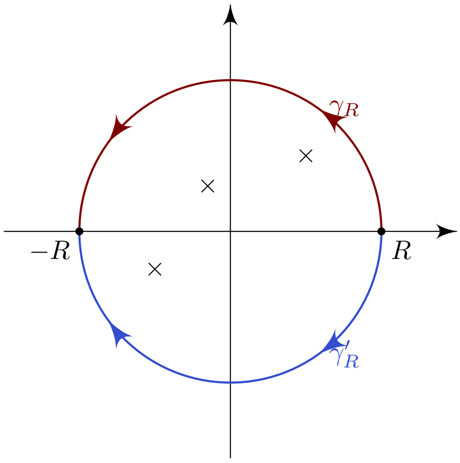
since γ_k encircles only one singularity, and we get a negative sign since γ_k is a clockwise contour. Then the result follows. □

4.2 Applications of the residue theorem

4.3 Further applications of the residue theorem using rectangular contours

4.4 Jordan's lemma

Lemma (Jordan's lemma)**.** Suppose that f is an analytic function, except for a finite number of singularities, and that $f(z) \to 0$ as $|z| \to \infty$.



Then for any real constant $\lambda > 0$, we have

$$\int_{\gamma_R} f(z)e^{i\lambda z}\ \mathrm{d}z \to 0$$

as $R \to \infty$, where γ_R is the semicircle of radius R in the upper half-plane.

For $\lambda < 0$, the same conclusion holds for the semicircular γ'_R in the lower half-plane.

Proof. The proof relies on the fact that for $\theta \in \left[0, \frac{\pi}{2}\right]$, we have

$$\sin\theta \geq \frac{2\theta}{\pi}.$$

So we get

$$\begin{aligned}
\left|\int_{\gamma_R} f(z)e^{i\lambda z}\ \mathrm{d}z\right| &= \left|\int_0^\pi f(Re^{i\theta})e^{i\lambda Re^{i\theta}} iRe^{i\theta}\ \mathrm{d}\theta\right|\\
&\leq R\int_0^\pi |f(Re^{i\theta})|\left|e^{i\lambda Re^{i\theta}}\right|\ \mathrm{d}\theta\\
&\leq 2R\sup_{z\in\gamma_R}|f(z)|\int_0^{\pi/2} e^{-\lambda R\sin\theta}\ \mathrm{d}\theta\\
&\leq 2R\sup_{z\in\gamma_R}|f(z)|\int_0^{\pi/2} e^{-2\lambda R\theta/\pi}\ \mathrm{d}\theta\\
&= \frac{\pi}{\lambda}(1 - e^{-\lambda R})\sup_{z\in\gamma_R}|f(z)|\\
&\to 0,
\end{aligned}$$

as required. Same for γ'_R when $\lambda < 0$. $\square$

5 Transform theory

5.1 Fourier transforms

5.2 Laplace transform

5.3 Elementary properties of the Laplace transform

Proposition.

(i) Linearity:
$$\mathcal{L}(\alpha f + \beta g) = \alpha\mathcal{L}(f) + \beta\mathcal{L}(g).$$

(ii) Translation:
$$\mathcal{L}(f(t - t_0)H(t - t_0)) = e^{-pt_0}\hat{f}(p).$$

(iii) Scaling:
$$\mathcal{L}(f(\lambda t)) = \frac{1}{\lambda}\hat{f}\left(\frac{p}{\lambda}\right),$$
where we require $\lambda > 0$ so that $f(\lambda t)$ vanishes for $t < 0$.

(iv) Shifting:
$$\mathcal{L}(e^{p_0 t}f(t)) = \hat{f}(p - p_0).$$

(v) Transform of a derivative:
$$\mathcal{L}(f'(t)) = p\hat{f}(p) - f(0).$$
Repeating the process,
$$\mathcal{L}(f''(t)) = p\mathcal{L}(f'(t)) - f'(0) = p^2\hat{f}(p) - pf(0) - f'(0),$$
and so on. This is the key fact for solving ODEs using Laplace transforms.

(vi) Derivative of a transform:
$$\hat{f}'(p) = \mathcal{L}(-tf(t)).$$
Of course, the point of this is not that we know what the derivative of $\hat{f}$ is. It is we know how to find the Laplace transform of $tf(t)$! For example, this lets us find the derivative of t^2 with ease.

In general,
$$\hat{f}^{(n)}(p) = \mathcal{L}((-t)^n f(t)).$$

(vii) Asymptotic limits
$$p\hat{f}(p) \to \begin{cases} f(0) & \text{as } p \to \infty \\ f(\infty) & \text{as } p \to 0 \end{cases},$$
where the second case requires f to have a limit at ∞.

Proof.

(v) We have

$$\int_0^\infty f'(t)e^{-pt}\,\mathrm{d}t = [f(t)e^{-pt}]_0^\infty + p\int_0^\infty f(t)e^{-pt}\,\mathrm{d}t = p\hat{f}(p) - f(0).$$

(vi) We have

$$\hat{f}(p) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t.$$

Differentiating with respect to p, we have

$$\hat{f}'(p) = -\int_0^\infty tf(t)e^{-pt}\,\mathrm{d}t.$$

(vii) Using (v), we know

$$p\hat{f}(p) = f(0) + \int_0^\infty f'(t)e^{-pt}\,\mathrm{d}t.$$

As $p \to \infty$, we know $e^{-pt} \to 0$ for all t. So $p\hat{f}(p) \to f(0)$. This proof looks dodgy, but is actually valid since f' grows no more than exponentially fast.

Similarly, as $p \to 0$, then $e^{-pt} \to 1$. So

$$p\hat{f}(p) \to f(0) + \int_0^\infty f'(t)\,\mathrm{d}t = f(\infty).$$

□

5.4 The inverse Laplace transform

Proposition. The inverse Laplace transform is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(p)e^{pt}\,\mathrm{d}p,$$

where c is a real constant such that the *Bromwich inversion contour* γ given by $\operatorname{Re} p = c$ lies to the *right* of all the singularities of $\hat{f}(p)$.

Proof. Since f has a Laplace transform, it grows no more than exponentially. So we can find a $c \in \mathbb{R}$ such that

$$g(t) = f(t)e^{-ct}$$

decays at infinity (and is zero for $t < 0$, of course). So g has a Fourier transform, and

$$\tilde{g}(\omega) = \int_{-\infty}^\infty f(t)e^{-ct}e^{-i\omega t}\,\mathrm{d}t = \hat{f}(c + i\omega).$$

Then we use the Fourier inversion formula to obtain

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(c + i\omega)e^{i\omega t}\,\mathrm{d}\omega.$$

So we make the substitution $p = c + i\omega$, and thus obtain

$$f(t)e^{-ct} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{f}(p)e^{(p-c)t}\,\mathrm{d}p.$$

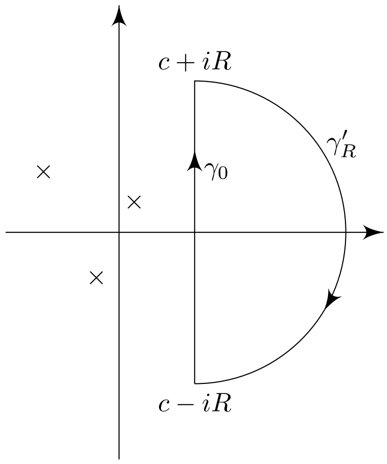
Multiplying both sides by e^{ct}, we get the result we were looking for (the requirement that c lies to the right of all singularities is to fix the "constant of integration" so that $f(t) = 0$ for all $t < 0$, as we will soon see). □

Proposition. In the case that $\hat{f}(p)$ has only a finite number of isolated singularities p_k for $k = 1, \cdots, n$, and $\hat{f}(p) \to 0$ as $|p| \to \infty$, then

$$f(t) = \sum_{k=1}^{n} \operatorname*{res}_{p=p_k} (\hat{f}(p)e^{pt})$$

for $t > 0$, and vanishes for $t < 0$.

Proof. We first do the case where $t < 0$, consider the contour $\gamma_0 + \gamma'_R$ as shown, which encloses no singularities.



Now if $\hat{f}(p) = o(|p|^{-1})$ as $|p| \to \infty$, then

$$\left| \int_{\gamma'_R} \hat{f}(p)e^{pt}\;\mathrm{d}p \right| \leq \pi R e^{ct} \sup_{p \in \gamma'_R} |\hat{f}(p)| \to 0 \text{ as } R \to \infty.$$

Here we used the fact $|e^{pt}| \leq e^{ct}$, which arises from $\mathrm{Re}(pt) \leq ct$, noting that $t < 0$.

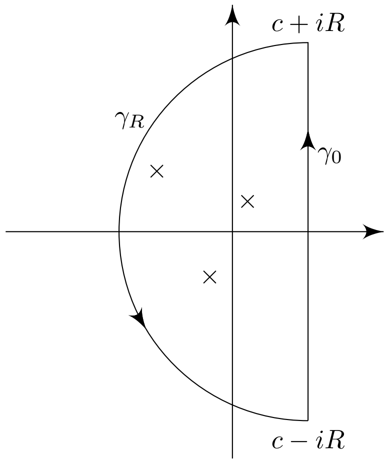
If $\hat{f}$ decays less rapidly at infinity, but still tends to zero there, the same result holds, but we need to use a slight modification of Jordan's lemma. So in either case, the integral

$$\int_{\gamma'_R} \hat{f}(p)e^{pt}\;\mathrm{d}p \to 0 \text{ as } R \to \infty.$$

Thus, we know $\int_{\gamma_0} \to \int_{\gamma}$. Hence, by Cauchy's theorem, we know $f(t) = 0$ for $t < 0$. This is in agreement with the requirement that functions with Laplace transform vanish for $t < 0$.

Here we see why γ must lie to the right of all singularities. If not, then the contour would encircle some singularities, and then the integral would no longer be zero.

When $t > 0$, we close the contour to the left.



This time, our γ *does* enclose some singularities. Since there are only finitely many singularities, we enclose all singularities for sufficiently large R. Once again, we get $\int_{\gamma_R} \to 0$ as $R \to \infty$. Thus, by the residue theorem, we know

$$\int_\gamma \hat{f}(p) e^{pt}\;\mathrm{d}p = \lim_{R\to\infty} \int_{\gamma_0} \hat{f}(p) e^{pt}\;\mathrm{d}p = 2\pi i \sum_{k=1}^n \operatorname*{res}_{p = p_k} (\hat{f}(p) e^{pt}).$$

So from the Bromwich inversion formula,

$$f(t) = \sum_{k=1}^n \operatorname*{res}_{p = p_k} (\hat{f}(p) e^{pt}),$$

as required. □

5.5 Solution of differential equations using the Laplace transform

5.6 The convolution theorem for Laplace transforms

Theorem (Convolution theorem)**.** The Laplace transform of a convolution is given by

$$\mathcal{L}(f * g)(p) = \hat{f}(p)\hat{g}(p).$$

Proof.

$$\mathcal{L}(f * g)(p) = \int_0^\infty \left(\int_0^t f(t - t')g(t')\;\mathrm{d}t'\right) e^{-pt}\;\mathrm{d}t$$

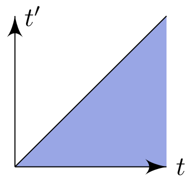
We change the order of integration in the (t, t') plane, and adjust the limits accordingly (see picture below)

$$= \int_0^\infty \left(\int_{t'}^\infty f(t - t')g(t') e^{-pt}\;\mathrm{d}t\right)\;\mathrm{d}t'$$

We substitute $u = t - t'$ to get

$$\begin{aligned} &= \int_0^\infty \left(\int_0^\infty f(u) g(t') e^{-pu} e^{-pt'}\;\mathrm{d}u\right)\;\mathrm{d}t' \\ &= \int_0^\infty \left(\int_0^\infty f(u) e^{-pu}\;\mathrm{d}u\right) g(t') e^{-pt'}\;\mathrm{d}t' \\ &= \hat{f}(p)\hat{g}(p). \end{aligned}$$

Note the limits of integration are correct since they both represent the region below



□