Part IB — Complex Methods
Theorems

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Analytic functions
Definition of an analytic function. Cauchy-Riemann equations. Analytic functions as conformal mappings; examples. Application to the solutions of Laplace's equation in various domains. Discussion of $\log z$ and z^a. [5]

Contour integration and Cauchy's Theorem
[*Proofs of theorems in this section will not be examined in this course.*]
Contours, contour integrals. Cauchy's theorem and Cauchy's integral formula. Taylor and Laurent series. Zeros, poles and essential singularities. [3]

Residue calculus
Residue theorem, calculus of residues. Jordan's lemma. Evaluation of definite integrals by contour integration. [4]

Fourier and Laplace transforms
Laplace transform: definition and basic properties; inversion theorem (proof not required); convolution theorem. Examples of inversion of Fourier and Laplace transforms by contour integration. Applications to differential equations. [4]

Contents

0 Introduction

1 Analytic functions

1.1 The complex plane and the Riemann sphere

1.2 Complex differentiation

Proposition (Cauchy-Riemann equations)**.** If $f = u + iv$ is differentiable, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proposition. Given a complex function $f = u + iv$, if u and v are real differentiable at a point z and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then f is differentiable at z.

1.3 Harmonic functions

Proposition. The real and imaginary parts of any analytic function are harmonic.

1.4 Multi-valued functions

1.5 Möbius map

Proposition. Möbius maps take circlines to circlines.

Proposition. Given six points $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{C}^*$, we can find a Möbius map which sends $\alpha \mapsto \alpha', \beta \mapsto \beta'$ and $\gamma \to \gamma'$.

1.6 Conformal maps

Proposition. A conformal map preserves the angles between intersecting curves.

1.7 Solving Laplace's equation using conformal maps

2 Contour integration and Cauchy's theorem

2.1 Contour and integrals

Proposition.

(i) We write $\gamma_1 + \gamma_2$ for the path obtained by joining γ_1 and γ_2. We have

$$\int_{\gamma_1+\gamma_2} f(z)\ \mathrm{d}z = \int_{\gamma_1} f(z)\ \mathrm{d}z + \int_{\gamma_2} f(z)\ \mathrm{d}z$$

Compare this with the equivalent result on the real line:

$$\int_a^c f(x)\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x + \int_b^c f(x)\ \mathrm{d}x.$$

(ii) Recall $-\gamma$ is the path obtained from reversing γ. Then we have

$$\int_{-\gamma} f(z)\ \mathrm{d}z = -\int_\gamma f(z)\ \mathrm{d}z.$$

Compare this with the real result

$$\int_a^b f(x)\ \mathrm{d}x = -\int_b^a f(x)\ \mathrm{d}x.$$

(iii) If γ is a contour from a to b in $\mathbb{C}$, then

$$\int_\gamma f'(z)\ \mathrm{d}z = f(b) - f(a).$$

This looks innocuous. This is just the fundamental theorem of calculus. However, there is some subtlety. This requires f to be differentiable at every point on γ. In particular, it must not cross a branch cut. For example, our previous example had $\log z$ as the antiderivative of $\frac{1}{z}$. However, this does not imply the integrals along different paths are the same, since we need to pick different branches of log for different paths, and things become messy.

(iv) Integration by substitution and by parts work exactly as for integrals on the real line.

(v) If γ has length L and $|f(z)|$ is bounded by M on γ, then

$$\left|\int_\gamma f(z)\ \mathrm{d}z\right| \leq LM.$$

This is since

$$\left|\int_\gamma f(z)\ \mathrm{d}z\right| \leq \int_\gamma |f(z)||\ \mathrm{d}z| \leq M\int_\gamma |\mathrm{d}z| = ML.$$

We will be using this result a lot later on.

2.2 Cauchy's theorem

Theorem (Cauchy's theorem)**.** If $f(z)$ is analytic in a simply-connected domain $\mathcal{D}$, then for every simple closed contour γ in $\mathcal{D}$, we have

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

2.3 Contour deformation

Proposition. Suppose that γ_1 and γ_2 are contours from a to b, and that f is analytic on the contours *and* between the contours. Then

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

2.4 Cauchy's integral formula

Theorem (Cauchy's integral formula)**.** Suppose that $f(z)$ is analytic in a domain $\mathcal{D}$ and that $z_0 \in \mathcal{D}$. Then

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

for any simple closed contour γ in $\mathcal{D}$ encircling z_0 anticlockwise.

Theorem (Liouville's theorem*)**.** Any bounded entire function is a constant.

3 Laurent series and singularities

3.1 Taylor and Laurent series

Proposition (Laurent series)**.** If f is analytic in an *annulus* $R_1 < |z - z_0| < R_2$, then it has a *Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

This is convergent within the annulus. Moreover, the convergence is uniform within compact subsets of the annulus.

3.2 Zeros

3.3 Classification of singularities

3.4 Residues

Proposition. At a *simple* pole, the residue is given by

$$\operatorname*{res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proposition. At a pole of order N, the residue is given by

$$\lim_{z \to z_0} \frac{1}{(N-1)!} \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}} (z - z_0)^N f(z).$$

Theorem. γ be an anticlockwise simple closed contour, and let f be analytic within γ except for an isolated singularity z_0. Then

$$\oint_\gamma f(z)\, \mathrm{d}z = 2\pi i a_{-1} = 2\pi i \operatorname*{res}_{z=z_0} f(z).$$

4 The calculus of residues

4.1 The residue theorem

Theorem (Residue theorem)**.** Suppose f is analytic in a simply-connected region except at a finite number of isolated singularities $z_1, \cdots, z_n$, and that a simple closed contour γ encircles the singularities anticlockwise. Then

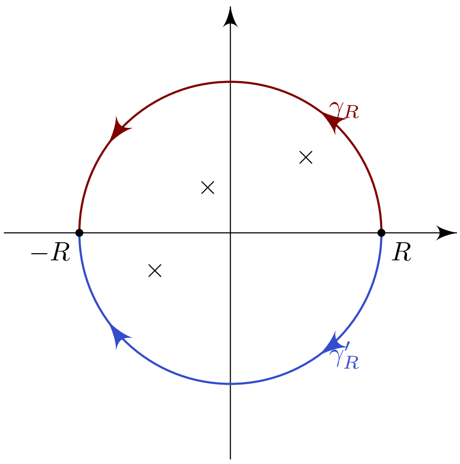
$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{z=z_k} f(z).$$

4.2 Applications of the residue theorem

4.3 Further applications of the residue theorem using rectangular contours

4.4 Jordan's lemma

Lemma (Jordan's lemma)**.** Suppose that f is an analytic function, except for a finite number of singularities, and that $f(z) \to 0$ as $|z| \to \infty$.



Then for any real constant $\lambda > 0$, we have

$$\int_{\gamma_R} f(z)e^{i\lambda z}\,\mathrm{d}z \to 0$$

as $R \to \infty$, where γ_R is the semicircle of radius R in the upper half-plane.

For $\lambda < 0$, the same conclusion holds for the semicircular γ_R' in the lower half-plane.

5 Transform theory

5.1 Fourier transforms

5.2 Laplace transform

5.3 Elementary properties of the Laplace transform

Proposition.

(i) Linearity:
$$\mathcal{L}(\alpha f + \beta g) = \alpha\mathcal{L}(f) + \beta\mathcal{L}(g).$$

(ii) Translation:
$$\mathcal{L}(f(t - t_0)H(t - t_0)) = e^{-pt_0}\hat{f}(p).$$

(iii) Scaling:
$$\mathcal{L}(f(\lambda t)) = \frac{1}{\lambda}\hat{f}\left(\frac{p}{\lambda}\right),$$
where we require $\lambda > 0$ so that $f(\lambda t)$ vanishes for $t < 0$.

(iv) Shifting:
$$\mathcal{L}(e^{p_0 t}f(t)) = \hat{f}(p - p_0).$$

(v) Transform of a derivative:
$$\mathcal{L}(f'(t)) = p\hat{f}(p) - f(0).$$
Repeating the process,
$$\mathcal{L}(f''(t)) = p\mathcal{L}(f'(t)) - f'(0) = p^2\hat{f}(p) - pf(0) - f'(0),$$
and so on. This is the key fact for solving ODEs using Laplace transforms.

(vi) Derivative of a transform:
$$\hat{f}'(p) = \mathcal{L}(-tf(t)).$$
Of course, the point of this is not that we know what the derivative of $\hat{f}$ is. It is we know how to find the Laplace transform of $tf(t)$! For example, this lets us find the derivative of t^2 with ease.

In general,
$$\hat{f}^{(n)}(p) = \mathcal{L}((-t)^n f(t)).$$

(vii) Asymptotic limits
$$p\hat{f}(p) \to \begin{cases} f(0) & \text{as } p \to \infty \\ f(\infty) & \text{as } p \to 0 \end{cases},$$
where the second case requires f to have a limit at ∞.

5.4 The inverse Laplace transform

Proposition. The inverse Laplace transform is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(p)e^{pt}\;\mathrm{d}p,$$

where c is a real constant such that the *Bromwich inversion contour* γ given by $\operatorname{Re} p = c$ lies to the *right* of all the singularities of $\hat{f}(p)$.

Proposition. In the case that $\hat{f}(p)$ has only a finite number of isolated singularities p_k for $k = 1, \cdots, n$, and $\hat{f}(p) \to 0$ as $|p| \to \infty$, then

$$f(t) = \sum_{k=1}^{n} \operatorname*{res}_{p=p_k} (\hat{f}(p)e^{pt})$$

for $t > 0$, and vanishes for $t < 0$.

5.5 Solution of differential equations using the Laplace transform

5.6 The convolution theorem for Laplace transforms

Theorem (Convolution theorem)**.** The Laplace transform of a convolution is given by

$$\mathcal{L}(f * g)(p) = \hat{f}(p)\hat{g}(p).$$