Complex Methods: Example Sheet 1

Part IB, Lent Term 2016

Dr R. E. Hunt

Cauchy–Riemann equations

1. (i) Where, if anywhere, in the complex plane are the following functions differentiable, and where are they analytic?

$$\operatorname{Im} z; \qquad |z|^2; \qquad \operatorname{sech} z.$$

(ii) Let $f(z) = z^5/|z|^4$, $z \neq 0$, $f(0) = 0$. Show that the real and imaginary parts of f satisfy the Cauchy–Riemann equations at $z = 0$, but that f is not differentiable at $z = 0$.

2. Find, as functions of z, complex analytic functions $f(z)$ whose real parts are the following:

(i) x (ii) xy (iii) $\sin x \cosh y$

(iv) $\log(x^2 + y^2)$ (v) $\dfrac{y}{(x+1)^2 + y^2}$ (vi) $\tan^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$

Deduce that the above functions are harmonic on appropriately-chosen domains, which you should specify.

3. By considering $w(z) = (i + z)/(i - z)$, show that $\phi(x, y) = \tan^{-1} \dfrac{2x}{x^2 + y^2 - 1}$ is harmonic.

4. Verify that the function $\phi(x, y) = e^x(x \cos y - y \sin y)$ is harmonic. Find its harmonic conjugate and, by considering $\nabla\phi$ or otherwise, determine the family of curves orthogonal to $\phi(x, y) = c$ for a given constant c.

Find an analytic function $f(z)$ such that $\operatorname{Re} f = \phi$. Can the expression $f(z) = \phi(z, 0)$ be used to determine $f(z)$ in general?

Branches of multi-valued functions

5. Show how the principal branch of $\log z$ can be used to define a branch of z^i which is single-valued on the domain $\mathscr{D} = \mathbb{C} \setminus \mathbb{R}^-$. Evaluate i^i for this branch.

Show, using polar coordinates, that the branch of z^i defined above maps $\mathscr{D}$ onto an annulus which is covered infinitely often.

How would your answers change, if at all, for a different branch?

6. How many branch points does $f(z) = [z(z + 1)]^{1/3}$ have? Draw some possible branch cuts in the complex plane and on the Riemann sphere.

Repeat for $f(z) = (z^2 + 1)^{1/2}$.

∗ Repeat also for $f(z) = [z(z + 1)(z + 2)]^{1/3}$ and $f(z) = [z(z + 1)(z + 2)(z + 3)]^{1/2}$.

7. Let $f(z) = (z^2 - 1)^{1/2}$, and consider two different branches of the function $f(z)$:

$f_1(z)$: branch cut $[-1, 1]$, $f_1(x) = +\sqrt{x^2 - 1}$ for real $x > 1$;

$f_2(z)$: branch cut $(-\infty, -1] \cup [1, \infty)$, $f_2(x) = +i\sqrt{1 - x^2}$ for real $x \in (-1, 1)$.

Find the limiting values of f_1 and f_2 above and below their respective branch cuts. Prove that f_1 is an odd function, i.e., $f_1(z) = -f_1(-z)$, and that f_2 is even.

Conformal mappings

8. How does the disc $|z-1|<1$ transform under the mapping $z \mapsto z^{-1}$?

Use the identity
$$\frac{z}{(z-1)^2} = \left(\frac{1}{1-z} - \frac{1}{2}\right)^2 - \frac{1}{4}$$
to show that the map $f(z) = z/(z-1)^2$ is a one-to-one conformal mapping of the disc $|z|<1$ onto the domain $\mathbb{C} \setminus \{x+iy : x \leqslant -\frac{1}{4},\ y=0\}$.

9. Find conformal mappings f_i of $\mathscr{U}_i$ onto $\mathscr{V}_i$ for each of the following cases. If the mapping is a composition of several functions, provide a sketch for each step. $\mathscr{D}$ denotes the unit disc $\{z : |z| < 1\}$.

(i) $\mathscr{U}_1 = \{z : \operatorname{Re} z < 0,\ -1 < \operatorname{Im} z < 1\}$, $\mathscr{V}_1 = \mathscr{D}$.

(ii) $\mathscr{U}_2 = \mathscr{D}$, $\mathscr{V}_2$ is the cut complex plane $\mathbb{C} \setminus \mathbb{R}^-$.

(iii) $\mathscr{U}_3$ is the angular sector $\{z : 0 < \arg z < \alpha\}$, $\mathscr{V}_3 = \{z : 0 < \operatorname{Im} z < 1\}$.

∗ (iv) $\mathscr{U}_4$ is the open region bounded between two circles $\{z : |z| < 1,\ |z+i| > \sqrt{2}\}$, $\mathscr{V}_4 = \mathscr{D}$.

Laplace's equation

10. Show that

$g(z) = e^z$ maps the strip $\mathscr{S} = \{z : 0 < \operatorname{Im} z < \pi\}$ onto the UHP $\{z : \operatorname{Im} z > 0\}$,

$h(z) = \sin z$ maps the half-strip $\mathscr{H} = \{z : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2},\ \operatorname{Im} z > 0\}$ onto the UHP.

Find a conformal map $f\colon \mathscr{H} \to \mathscr{S}$. Hence find a function $\phi(x,y)$ which is harmonic on the half-strip $\mathscr{H}$ with the following limiting values on its boundary $\partial\mathscr{H}$:
$$\phi(x,y) = \begin{cases} 0 & \text{on } \partial\mathscr{H} \text{ in the LHP } (x<0), \\ 1 & \text{on } \partial\mathscr{H} \text{ in the RHP } (x>0). \end{cases}$$

Give ϕ as a function of x and y. Is there only one such function?

11. Using conformal mapping(s), find a solution to Laplace's equation in the upper half-plane $\{z : \operatorname{Im} z > 0\}$ with boundary conditions
$$\phi(x,0) = \begin{cases} 1 & x \in [-1,1], \\ 0 & \text{otherwise.} \end{cases}$$

[*Find a map f of the upper half-plane onto itself that makes the boundary conditions easier to deal with.*]

Comments on or corrections to this problem sheet are very welcome and may be sent to reh10@cam.ac.uk.

Complex Methods: Example Sheet 2

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Taylor and Laurent series

1. Find the first two non-vanishing coefficients in the Taylor expansion about the origin of the following functions, assuming principal branches when there is a choice. You may make use of standard series expansions for $\log(1+z)$, etc.

$$\text{(i)}\quad z/\log(1+z) \qquad \text{(ii)}\quad (\cos z)^{1/2}-1 \qquad \text{(iii)}\quad \log(1+e^z) \qquad \text{(iv)}\quad e^{e^z}.$$

State the range of values of z for which each series converges.

How would your answers differ if you assumed branches different from the principal branch?

2. Let a, b be complex constants with $0<|a|<|b|$. Use partial fractions to find the Laurent expansions of $1/\{(z-a)(z-b)\}$ about $z=0$ in each of the regions $|z|<|a|$, $|a|<|z|<|b|$ and $|z|>|b|$.

3. Find the first three terms of the Laurent expansion of $f(z)=\operatorname{cosec}^2 z$ valid for $0<|z|<\pi$.

$*$ Show that the function $h(z)=f(z)-z^{-2}-(z+\pi)^{-2}-(z-\pi)^{-2}$ has only removable singularities in $|z|<2\pi$. Explain how to remove them to obtain a function $H(z)$ analytic in that region. Find a Taylor series for $H(z)$ about the origin and explain why it must be convergent in $|z|<2\pi$. Hence, or otherwise, find the three non-zero central terms of the Laurent expansion of $f(z)$ valid for $\pi<|z|<2\pi$.

4. Write down the location and type of each of the singularities of the following functions:

$$\text{(i)}\quad \frac{1}{z^3(z-1)^2} \qquad \text{(ii)}\quad \tan z \qquad \text{(iii)}\quad z\coth z \qquad \text{(iv)}\quad \frac{e^z-e}{(1-z)^3}$$

$$\text{(v)}\quad \exp(\tan z) \qquad \text{(vi)}\quad \sinh\frac{z}{z^2-1} \qquad \text{(vii)}\quad \log(1+e^z) \qquad \text{(viii)}\quad \tan(z^{-1}).$$

Integration and residues

5. Evaluate $\oint_\gamma \bar{z}\,\mathrm{d}z$ when γ is the circle $|z|=1$, and when γ is the circle $|z-1|=1$.

6. (i) Show that if $f(z)$ is analytic, then the residue of $f(z)/(z-z_0)$ at $z=z_0$ is $f(z_0)$.

(ii) Show that if $1/f(z)$ has a simple pole at $z=z_0$, then its residue at $z=z_0$ is $1/f'(z_0)$.

(iii) Show that if $h(z)$ has a simple zero at $z=z_0$ and $g(z)$ is analytic and non-zero, the residue of $g(z)/h(z)$ at $z=z_0$ is $g(z_0)/h'(z_0)$.

(iv) Prove the formula for the residue of a function $f(z)$ that has a pole of order N at $z=z_0$:

$$\lim_{z\to z_0}\left\{\frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\big((z-z_0)^N f(z)\big)\right\}.$$

7. Evaluate, using Cauchy's theorem or the residue theorem,

$$\text{(i)}\quad \oint_{\gamma_1}\frac{\mathrm{d}z}{1+z^2} \qquad \text{(ii)}\quad \oint_{\gamma_2}\frac{\mathrm{d}z}{1+z^2} \qquad \text{(iii)}\quad \oint_{\gamma_3}\frac{e^z\cos z\,\mathrm{d}z}{(1+z^2)\sin z} \qquad *\,\text{(iv)}\quad \oint_{\gamma_4}\frac{z^3e^{1/z}\,\mathrm{d}z}{1+z}$$

where γ_1 is the ellipse $x^2+4y^2=1$, γ_2 is the circle $x^2+y^2=2$, γ_3 is the circle $|z-(2+i)|=\sqrt{2}$ and γ_4 is the circle $|z|=2$.

8. By integrating the function $z^n(z-a)^{-1}(z-a^{-1})^{-1}$ around the unit circle in the z-plane (where a is real, $a > 1$, and n is a non-negative integer), evaluate

$$\int_0^{2\pi} \frac{\cos n\theta}{1 - 2a\cos\theta + a^2}\,\mathrm{d}\theta.$$

The calculus of residues

9. Evaluate $\displaystyle\lim_{R\to\infty}\int_{-R}^{R}\frac{x\,\mathrm{d}x}{1+x+x^2}$. Why is the limit here rather than just the integral $\int_{-\infty}^{\infty}$?

10. By integrating around a keyhole contour, show that

$$\int_0^\infty \frac{x^{a-1}\,\mathrm{d}x}{1+x} = \frac{\pi}{\sin(\pi a)} \qquad (0 < a < 1).$$

Explain why the given restrictions on the value of a are necessary.

$*$ **11.** By integrating around a contour involving the real axis and the line $z = re^{2\pi i/n}$, evaluate

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^n} \qquad (n \geqslant 2).$$

Check (by change of variable) that your answer agrees with that of the previous question.

12. Establish the following:

$$\text{(i)}\quad \int_0^\infty \frac{\cos x}{(1+x^2)^3}\,\mathrm{d}x = \frac{7\pi}{16e} \qquad \text{(ii)}\quad \int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \frac{\pi}{2} \qquad \text{(iii)}\quad \int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x = 0.$$

[*For part (iii), integrate* $(\log z)^2/(1+z^2)$ *around a keyhole, or* $\log z/(1+z^2)$ *along the real axis (or both). What goes wrong if you integrate* $\log z/(1+z^2)$ *around a keyhole?*]

$*$ **13.** Let $P(z)$ be a non-constant polynomial. Consider the contour integral

$$I = \oint_\gamma \frac{P'(z)}{P(z)}\,\mathrm{d}z.$$

Show that, if γ is a contour that encloses no zeros of P, then $I = 0$.

Evaluate the limit of I as $R \to \infty$, where γ is the circle $|z| = R$, and deduce that P has at least one zero in the complex plane.

14. By considering the integral of $f(z) = \cot z/(z^2 + \pi^2 a^2)$ around a suitable large contour, prove that, provided ia is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a).$$

By considering a similar integral prove also that, if a is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

Find an expression for $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}$ and take the limit as $a \to 0$ to deduce the value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$.

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Complex Methods: Example Sheet 3

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Fourier Transforms

1. Let

$$f(x) = \begin{cases} 1 & |x| < \tfrac{1}{2}a, \\ 0 & \text{otherwise}; \end{cases} \qquad g(x) = \begin{cases} a - |x| & |x| < a, \\ 0 & \text{otherwise}. \end{cases}$$

Show that

$$\widetilde{f}(k) = \frac{2}{k} \sin \frac{ak}{2} \quad \text{and} \quad \widetilde{g}(k) = \frac{4}{k^2} \sin^2 \frac{ak}{2}.$$

Verify by contour integration the inversion formula for $f(x)$, including the value at $x = \frac{1}{2}a$. What is the convolution of f with itself?

2. Using the results of the previous question and Parseval's identity, evaluate the integrals

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4}\,\mathrm{d}x.$$

3. By using the relationship between the Fourier transform and its inverse, show that for real a and b with $a > 0$,

$$e^{-a|t|} = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2}\, e^{i\omega t}\,\mathrm{d}\omega \quad \text{and} \quad e^{-at} \sin bt\, H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b}{(i\omega + a)^2 + b^2}\, e^{i\omega t}\,\mathrm{d}\omega$$

where $H(t)$ is the Heaviside step function. What are the values of the integrals when $a < 0$? What happens when $a = 0$?

4. Show that the convolution of the function $e^{-|x|}$ with itself is given by $f(x) = (1 + |x|)e^{-|x|}$. Use the convolution theorem to show that

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(1 + k^2)^2}\,\mathrm{d}k$$

and verify this result by contour integration.

$*$ **5.** Suppose that $f(x)$ has period 2π and let $g(x) = f(x)e^{-a|x|}$ where $a > 0$. Show that the Fourier Transform of g is given by

$$\widetilde{g}(k) = \frac{F(k - ia)}{1 - e^{-2\pi i(k - ia)}} - \frac{F(k + ia)}{1 - e^{-2\pi i(k + ia)}}$$

where $F(k) = \int_0^{2\pi} f(x)e^{-ikx}\,\mathrm{d}x$.

Assuming that F is analytic, sketch the locations of the singularities of $\widetilde{g}$ in the complex k-plane. Further assuming that F decays sufficiently quickly at infinity, use the Fourier inversion theorem and a suitable contour to show that

$$g(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n) e^{(in - a)x}$$

for $x > 0$ and derive a similar result when $x < 0$.

Deduce that

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n) e^{inx}.$$

[*This shows how the Fourier transform representation of a function reduces to a Fourier series if the function is periodic.*]

Laplace Transforms

6. Starting from the Laplace transform of 1 (namely p^{-1}), and using only standard properties of the Laplace transform (shifting, etc.), find the Laplace transforms of the following functions: (i) e^{-2t} (ii) $t^3 e^{-3t}$ (iii) $e^{3t} \sin 4t$ (iv) $e^{-4t} \cosh 2t$.

7. Using partial fractions and expressions for the Laplace transforms of elementary functions, find the inverse Laplace transform of $\widehat{f}(p) = (p+3)/\{(p-2)(p^2+1)\}$. Verify this result using the Bromwich inversion formula.

8. Use Laplace transforms to solve the differential equation

$$y''' - 3y'' + 3y' - y = t^2 e^t$$

with initial conditions $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$.

9. Solve the *integral equation* $f(t) + 4 \int_0^t (t-\tau) f(\tau) \, \mathrm{d}\tau = t$ for the unknown function *f*. Verify your solution.

10. Solve the system of differential equations $\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 & 10 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ with $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ at $t = 0$.

$*$ **11.** The zeroth order Bessel function $J_0(t)$ satisfies the differential equation

$$t J_0'' + J_0' + t J_0 = 0$$

with $J_0(0) = 1$ (and $J_0'(0) = 0$ from the equation). Find the Laplace transform of J_0 and deduce that $\int_0^\infty J_0(t) \, \mathrm{d}t = 1$. Find the convolution of J_0 with itself.

12. Use Laplace transforms to solve the heat equation $\partial T/\partial t = \partial^2 T/\partial x^2$ with boundary conditions $T(x, 0) = 3 \sin 2\pi x$ $(0 < x < 1)$, $T(0, t) = T(1, t) = 0$ $(t > 0)$.

13. Using the equality $\int_0^\infty e^{-x^2} \mathrm{d}x = \frac{1}{2}\sqrt{\pi}$, find the Laplace transform of $f(t) = t^{-1/2}$. By integrating around a Bromwich keyhole contour, verify the inversion formula for $f(t)$. What is the Laplace transform of $t^{1/2}$?

$*$ **14.** The gamma and beta functions are defined for $z, w \in \mathbb{C}$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t \quad \text{and} \quad B(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} \, \mathrm{d}t$$

when $\operatorname{Re} z, \operatorname{Re} w > 0$ (and by analytic continuation elsewhere). Show that $\Gamma(z+1) = z\Gamma(z)$ and hence that $\Gamma(n+1) = n!$ if n is a positive integer. Using the previous question, write down the value of $\Gamma(\frac{1}{2})$.

For a fixed value of z, find the Laplace transform of $f(t) = t^{z-1}$ in terms of $\Gamma(z)$. Find the Laplace transform of the convolution $t^{z-1} * t^{w-1}$. Hence establish that

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$