Part IB — Complex Analysis
Theorems with proof

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Analytic functions
Complex differentiation and the Cauchy–Riemann equations. Examples. Conformal mappings. Informal discussion of branch points, examples of $\log z$ and z^c. [3]

Contour integration and Cauchy's theorem
Contour integration (for piecewise continuously differentiable curves). Statement and proof of Cauchy's theorem for star domains. Cauchy's integral formula, maximum modulus theorem, Liouville's theorem, fundamental theorem of algebra. Morera's theorem. [5]

Expansions and singularities
Uniform convergence of analytic functions; local uniform convergence. Differentiability of a power series. Taylor and Laurent expansions. Principle of isolated zeros. Residue at an isolated singularity. Classification of isolated singularities. [4]

The residue theorem
Winding numbers. Residue theorem. Jordan's lemma. Evaluation of definite integrals by contour integration. Rouché's theorem, principle of the argument. Open mapping theorem. [4]

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0 Introduction

1 Complex differentiation

1.1 Differentiation

Proposition. Let f be defined on an open set $U \subseteq \mathbb{C}$. Let $w = c + id \in U$ and write $f = u + iv$. Then f is complex differentiable at w if and only if u and v, viewed as a real function of two real variables, are differentiable at (c, d), *and*

$$\begin{aligned} u_x &= v_y, \\ u_y &= -v_x. \end{aligned}$$

These equations are the *Cauchy–Riemann equations*. In this case, we have

$$f'(w) = u_x(c, d) + iv_x(c, d) = v_y(c, d) - iu_y(c, d).$$

Proof. By definition, f is differentiable at w with $f'(w) = p + iq$ if and only if

$$\lim_{z \to w} \frac{f(z) - f(w) - (p + iq)(z - w)}{z - w} = 0. \qquad (\dagger)$$

If $z = x + iy$, then

$$(p + iq)(z - w) = p(x - c) - q(y - d) + i(q(x - c) + p(y - d)).$$

So, breaking into real and imaginary parts, we know $(\dagger)$ holds if and only if

$$\lim_{(x,y) \to (c,d)} \frac{u(x, y) - u(c, d) - (p(x - c) - q(y - d))}{\sqrt{(x - c)^2 + (y - d)^2}} = 0$$

and

$$\lim_{(x,y) \to (c,d)} \frac{v(x, y) - v(c, d) - (q(x - c) + p(y - d))}{\sqrt{(x - c)^2 + (y - d)^2}} = 0.$$

Comparing this to the definition of the differentiability of a real-valued function, we see this holds exactly if u and v are differentiable at (c, d) with

$$Du|_{(c,d)} = (p, -q), \quad Dv|_{(c,d)} = (q, p).$$ □

1.2 Conformal mappings

Theorem (Riemann mapping theorem)**.** Let $\mathcal{U} \subseteq \mathbb{C}$ be the bounded domain enclosed by a simple closed curve, or more generally any simply connected domain not equal to all of $\mathbb{C}$. Then $\mathcal{U}$ is conformally equivalent to $D = \{z : |z| < 1\} \subseteq \mathbb{C}$.

1.3 Power series

Proposition. The uniform limit of continuous functions is continuous.

Proposition (Weierstrass M-test)**.** For a sequence of functions f_n, if we can find $(M_n) \subseteq \mathbb{R}_{>0}$ such that $|f_n(x)| < M_n$ for all x in the domain, then $\sum M_n$ converges implies $\sum f_n(x)$ converges uniformly on the domain.

Proposition. Given any constants $\{c_n\}_{n \geq 0} \subseteq \mathbb{C}$, there is a unique $R \in [0, \infty]$ such that the series $z \mapsto \sum_{n=0}^{\infty} c_n(z - a)^n$ converges absolutely if $|z - a| < R$ and diverges if $|z - a| > R$. Moreover, if $0 < r < R$, then the series converges uniformly on $\{z : |z - a| < r\}$. This R is known as the *radius of convergence*.

Theorem. Let
$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$
be a power series with radius of convergence $R > 0$. Then

(i) f is holomorphic on $B(a; R) = \{z : |z - a| < R\}$.

(ii) $f'(z) = \sum n c_n (z-1)^{n-1}$, which also has radius of convergence R.

(iii) Therefore f is infinitely complex differentiable on $B(a; R)$. Furthermore,
$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Proof. Without loss of generality, take $a = 0$. The third part obviously follows from the previous two, and we will prove the first two parts simultaneously. We would like to first prove that the derivative series has radius of convergence R, so that we can freely happily manipulate it.

Certainly, we have $|nc_n| \geq |c_n|$. So by comparison to the series for f, we can see that the radius of convergence of $\sum n c_n z^{n-1}$ is at most R. But if $|z| < \rho < R$, then we can see
$$\frac{|nc_n z^{n-1}|}{|c_n \rho^{n-1}|} = n \left|\frac{z}{\rho}\right|^{n-1} \to 0$$
as $n \to \infty$. So by comparison to $\sum c_n \rho^{n-1}$, which converges, we see that the radius of convergence of $\sum n c_n z^{n-1}$ is at least ρ. So the radius of convergence must be exactly R.

Now we want to show f really is differentiable with that derivative. Pick z, w such that $|z|, |w| \leq \rho$ for some $\rho < R$ as before.

Define a new function
$$\varphi(z, w) = \sum_{n=1}^{\infty} c_n \sum_{j=0}^{n-1} z^j w^{n-1-j}.$$

Noting
$$\left| c_n \sum_{j=0}^{n-1} z^j w^{n-1-j} \right| \leq n |c_n| \rho^n,$$
we know the series defining φ converges uniformly on $\{|z| \leq \rho, |w| < \rho\}$, and hence to a continuous limit.

If $z \neq w$, then using the formula for the (finite) geometric series, we know
$$\varphi(z, w) = \sum_{n=1}^{\infty} c_n \left(\frac{z^n - w^n}{z - w}\right) = \frac{f(z) - f(w)}{z - w}.$$

On the other hand, if $z = w$, then
$$\varphi(z, z) = \sum_{n=1}^{\infty} c_n n z^{n-1}.$$

Since φ is continuous, we know

$$\lim_{w \to z} \frac{f(z) - f(w)}{z - w} \to \sum_{n=1}^{\infty} c_n n z^{n-1}.$$

So $f'(z) = \varphi(z, z)$ as claimed. Then (iii) follows from (i) and (ii) directly. □

Corollary. Given a power series

$$f(z) = \sum_{n \geq 0} c_n (z - a)^n$$

with radius of convergence $R > 0$, and given $0 < \varepsilon < R$, if f vanishes on $B(a, \varepsilon)$, then f vanishes identically.

Proof. If f vanishes on $B(a, \varepsilon)$, then all its derivatives vanish, and hence the coefficients all vanish. So it is identically zero. □

1.4 Logarithm and branch cuts

Proposition. On $\{z \in \mathbb{C} : z \not\in \mathbb{R}_{\leq 0}\}$, the principal branch $\log : U \to \mathbb{C}$ is holomorphic function. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}z} \log z = \frac{1}{z}.$$

If $|z| < 1$, then

$$\log(1 + z) = \sum_{n \geq 1} (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots.$$

Proof. The logarithm is holomorphic since it is a local inverse to a holomorphic function. Since $e^{\log z} = z$, the chain rule tells us $\frac{\mathrm{d}}{\mathrm{d}z}(\log z) = \frac{1}{z}$.

To show that $\log(1 + z)$ is indeed given by the said power series, note that the power series does have a radius of convergence 1 by, say, the ratio test. So by the previous result, it has derivative

$$1 - z + z^2 + \cdots = \frac{1}{1 + z}.$$

Therefore, $\log(1 + z)$ and the claimed power series have equal derivative, and hence coincide up to a constant. Since they agree at $z = 0$, they must in fact be equal. □

2 Contour integration

2.1 Basic properties of complex integration

Lemma. Suppose $f : [a, b] \to \mathbb{C}$ is continuous (and hence integrable). Then

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \le (b-a)\sup_t |f(t)|$$

with equality if and only if f is constant.

Proof. We let

$$\theta = \arg\left(\int_a^b f(t)\,\mathrm{d}t\right),$$

and

$$M = \sup_t |f(t)|.$$

Then we have

$$\begin{aligned}\left|\int_a^b f(t)\,\mathrm{d}t\right| &= \int_a^b e^{-i\theta} f(t)\,\mathrm{d}t\\ &= \int_a^b \mathrm{Re}(e^{-i\theta} f(t))\,\mathrm{d}t\\ &\le (b-a)M,\end{aligned}$$

with equality if and only if $|f(t)| = M$ and $\arg f(t) = \theta$ for all t, i.e. f is constant. □

Theorem (Fundamental theorem of calculus)**.** Let $f : U \to \mathbb{C}$ be continuous with antiderivative F. If $\gamma : [a, b] \to U$ is piecewise C^1-smooth, then

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Proof. We have

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b (F\circ\gamma)'(t)\,\mathrm{d}t.$$

Then the result follows from the usual fundamental theorem of calculus, applied to the real and imaginary parts separately. □

2.2 Cauchy's theorem

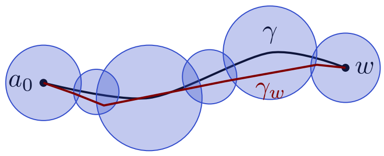
Proposition. Let $U \subseteq \mathbb{C}$ be a domain (i.e. path-connected non-empty open set), and $f : U \to \mathbb{C}$ be continuous. Moreover, suppose

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for any closed piecewise C^1-smooth path γ in U. Then f has an antiderivative.

Proof. Pick our favorite $a_0 \in U$. For $w \in U$, we choose a path $\gamma_w : [0,1] \to U$ such that $\gamma_w(0) = a_0$ and $\gamma_w(1) = w$.

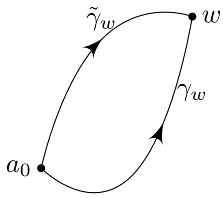
We first go through some topological nonsense to show we can pick γ_w such that this is piecewise C^1. We already know a *continuous* path $\gamma : [0,1] \to U$ from a_0 to w exists, by definition of path connectedness. Since U is open, for all x in the image of γ, there is some $\varepsilon(x) > 0$ such that $B(x, \varepsilon(x)) \subseteq U$. Since the image of γ is compact, it is covered by finitely many such balls. Then it is trivial to pick a piecewise straight path living inside the union of these balls, which is clearly piecewise smooth.



We thus define

$$F(w) = \int_{\gamma_w} f(z)\,\mathrm{d}z.$$

Note that this $F(w)$ is independent of the choice of γ_w, by our hypothesis on f — given another choice $\tilde{\gamma}_w$, we can form the new path $\gamma_w * (-\tilde{\gamma}_w)$, namely the path obtained by concatenating γ_w with $-\tilde{\gamma}_w$.



This is a closed piecewise C^1-smooth curve. So

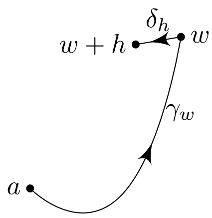
$$\int_{\gamma_w * (-\tilde{\gamma}_w)} f(z)\,\mathrm{d}z = 0.$$

The left hand side is

$$\int_{\gamma_w} f(z)\,\mathrm{d}z + \int_{-\tilde{\gamma}_w} f(z)\,\mathrm{d}z = \int_{\gamma_w} f(z)\,\mathrm{d}z - \int_{\tilde{\gamma}_w} f(z)\,\mathrm{d}z.$$

So the two integrals agree.

Now we need to check that F is complex differentiable. Since U is open, we can pick $\theta > 0$ such that $B(w; \varepsilon) \subseteq U$. Let δ_h be the radial path in $B(w, \varepsilon)$ from W to $w + h$, with $|h| < \varepsilon$.



Now note that $\gamma_w * \delta_h$ is a path from a_0 to $w + h$. So

$$\begin{aligned}F(w+h) &= \int_{\gamma_w * \delta_h} f(z)\,\mathrm{d}z\\ &= F(w) + \int_{\delta_h} f(z)\,\mathrm{d}z\\ &= F(w) + hf(w) + \int_{\delta_h} (f(z) - f(w))\,\mathrm{d}z.\end{aligned}$$

Thus, we know

$$\begin{aligned}\left|\frac{F(w+h) - F(w)}{h} - f(w)\right| &\leq \frac{1}{|h|}\left|\int_{\delta_h} f(z) - f(w)\,\mathrm{d}z\right|\\ &\leq \frac{1}{|h|}\operatorname{length}(\delta_h)\sup_{\delta_h}|f(z) - f(w)|\\ &= \sup_{\delta_h}|f(z) - f(w)|.\end{aligned}$$

Since f is continuous, as $h \to 0$, we know $f(z) - f(w) \to 0$. So F is differentiable with derivative f. ☐

Proposition. If U is a star domain, and $f : U \to \mathbb{C}$ is continuous, and if

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

for all triangles $T \subseteq U$, then f has an antiderivative on U.

Proof. As before, taking $\gamma_w = [a_0, w] \subseteq U$ if U is star-shaped about a_0. ☐

Theorem (Cauchy's theorem for a triangle)**.** Let U be a domain, and let $f : U \to \mathbb{C}$ be holomorphic. If $T \subseteq U$ is a triangle, then $\int_{\partial T} f(z)\,\mathrm{d}z = 0$.

Corollary (Convex Cauchy)**.** If U is a convex or star-shaped domain, and $f : U \to \mathbb{C}$ is holomorphic, then for *any* closed piecewise C^1 paths $\gamma \in U$, we must have

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof of corollary. If f is holomorphic, then Cauchy's theorem says the integral over any triangle vanishes. If U is star shaped, our proposition says f has an antiderivative. Then the fundamental theorem of calculus tells us the integral around any closed path vanishes. ☐

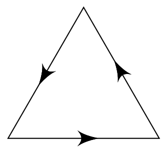
Proof of Cauchy's theorem for a triangle. Fix a triangle T. Let

$$\eta = \left|\int_{\partial T} f(z)\,\mathrm{d}z\right|,\quad \ell = \operatorname{length}(\partial T).$$

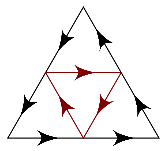
The idea is to show to bound η by ε, for every $\varepsilon > 0$, and hence we must have $\eta = 0$. To do so, we subdivide our triangles.

Before we start, it helps to motivate the idea of subdividing a bit. By subdividing the triangle further and further, we are focusing on a smaller and smaller region of the complex plane. This allows us to study how the integral behaves locally. This is helpful since we are given that f is holomorphic, and holomorphicity is a local property.

We start with $T = T^0$:



We then add more lines to get $T_a^0, T_b^0, T_c^0, T_d^0$ (it doesn't really matter which is which).



We orient the middle triangle by the anti-clockwise direction. Then we have

$$\int_{\partial T^0} f(z)\,\mathrm{d}z = \sum_{a,b,c,d} \int_{\partial T^0_\bullet} f(z)\,\mathrm{d}z,$$

since each internal edge occurs twice, with opposite orientation.

For this to be possible, if $\eta = \left|\int_{\partial T^0} f(z)\,\mathrm{d}z\right|$, then there must be some subscript in $\{a, b, c, d\}$ such that

$$\left|\int_{\partial T^0_\bullet} f(z)\,\mathrm{d}z\right| \geq \frac{\eta}{4}.$$

We call this $T^0_\bullet = T^1$. Then we notice ∂T^1 has length

$$\text{length}(\partial T^1) = \frac{\ell}{2}.$$

Iterating this, we obtain triangles

$$T^0 \supseteq T^1 \supseteq T^2 \supseteq \cdots$$

such that

$$\left|\int_{\partial T^i} f(z)\,\mathrm{d}z\right| \geq \frac{\eta}{4^i}, \quad \text{length}(\partial T^i) = \frac{\ell}{2^i}.$$

Now we are given a nested sequence of closed sets. By IB Metric and Topological Spaces (or IB Analysis II), there is some $z_0 \in \bigcap_{i \geq 0} T^i$.

Now fix an $\varepsilon > 0$. Since f is holomorphic at z_0, we can find a $\delta > 0$ such that

$$|f(w) - f(z_0) - (w - z_0)f'(z_0)| \leq \varepsilon |w - z_0|$$

whenever $|w - z_0| < \delta$. Since the diameters of the triangles are shrinking each time, we can pick an n such that $T^n \subseteq B(z_0, \varepsilon)$. We're almost there. We just need to do one last thing that is slightly sneaky. Note that

$$\int_{\partial T^n} 1\,\mathrm{d}z = 0 = \int_{\partial T^n} z\,\mathrm{d}z,$$

since these functions certainly do have anti-derivatives on T^n. Therefore, noting that $f(z_0)$ and $f'(z_0)$ are just constants, we have

$$\begin{aligned}\left|\int_{\partial T^n} f(z)\;\mathrm{d}z\right| &= \left|\int_{\partial T^n} (f(z) - f(z_0) - (z - z_0)f'(z_0))\;\mathrm{d}z\right|\\ &\leq \int_{\partial T^n} |f(z) - f(z_0) - (z - z_0)f'(z_0)|\;\mathrm{d}z\\ &\leq \text{length}(\partial T^n)\varepsilon \sup_{z \in \partial T^n} |z - z_0|\\ &\leq \varepsilon\, \text{length}(\partial T^n)^2,\end{aligned}$$

where the last line comes from the fact that $z_0 \in T^n$, and the distance between any two points in the triangle cannot be greater than the perimeter of the triangle. Substituting our formulas for these in, we have

$$\frac{\eta}{4^n} \leq \frac{1}{4^n}\ell^2\varepsilon.$$

So

$$\eta \leq \ell^2\varepsilon.$$

Since ℓ is fixed and ε was arbitrary, it follows that we must have $\eta = 0$. □

2.3 The Cauchy integral formula

Theorem (Cauchy integral formula). Let U be a domain, and $f : U \to \mathbb{C}$ be holomorphic. Suppose there is some $\overline{B(z_0; r)} \subseteq U$ for some z_0 and $r > 0$. Then for all $z \in B(z_0; r)$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial \overline{B(z_0; r)}} \frac{f(w)}{w - z}\;\mathrm{d}w.$$

Proof. Since U is open, there is some $\delta > 0$ such that $\overline{B(z_0; r + \delta)} \subseteq U$. We define $g : B(z_0; r + \delta) \to \mathbb{C}$ by

$$g(w) = \begin{cases} \frac{f(w) - f(z)}{w - z} & w \neq z\\ f'(z) & w = z\end{cases},$$

where we have *fixed* $z \in B(z_0; r)$ as in the statement of the theorem. Now note that g is holomorphic as a function of $w \in B(z_0, r + \delta)$, except perhaps at $w = z$. But since f is holomorphic, by definition g is continuous everywhere on $B(z_0, r + \delta)$. So the previous result says

$$\int_{\partial \overline{B(z_0; r)}} g(w)\;\mathrm{d}w = 0.$$

This is exactly saying that

$$\int_{\partial \overline{B(z_0; r)}} \frac{f(w)}{w - z}\;\mathrm{d}w = \int_{\partial \overline{B(z_0; r)}} \frac{f(z)}{w - z}\;\mathrm{d}w.$$

We now rewrite

$$\frac{1}{w-z} = \frac{1}{w-z_0} \cdot \frac{1}{1 - \left(\frac{z-z_0}{w-z_0}\right)} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}.$$

Note that this sum converges uniformly on $\partial\overline{B(z_0; r)}$ since

$$\left|\frac{z-z_0}{w-z_0}\right| < 1$$

for w on this circle.

By uniform convergence, we can exchange summation and integration. So

$$\int_{\partial\overline{B(z_0;r)}} \frac{f(w)}{w-z} \;\mathrm{d}w = \sum_{n=0}^{\infty} \int_{\partial\overline{B(z_0,r)}} f(z)\frac{(z-z_0)^n}{(w-z_0)^{n+1}} \;\mathrm{d}w.$$

We note that $f(z)(z-z_0)^n$ is just a constant, and that we have previously proven

$$\int_{\partial\overline{B(z_0;r)}} (w-z_0)^k \;\mathrm{d}w = \begin{cases} 2\pi i & k=-1 \\ 0 & k \neq -1 \end{cases}.$$

So the right hand side is just $2\pi i f(z)$. So done. □

Corollary (Local maximum principle)**.** Let $f : B(z,r) \to \mathbb{C}$ be holomorphic. Suppose $|f(w)| \leq |f(z)|$ for all $w \in B(z;r)$. Then f is constant. In other words, a non-constant function cannot achieve an interior local maximum.

Proof. Let $0 < \rho < r$. Applying the Cauchy integral formula, we get

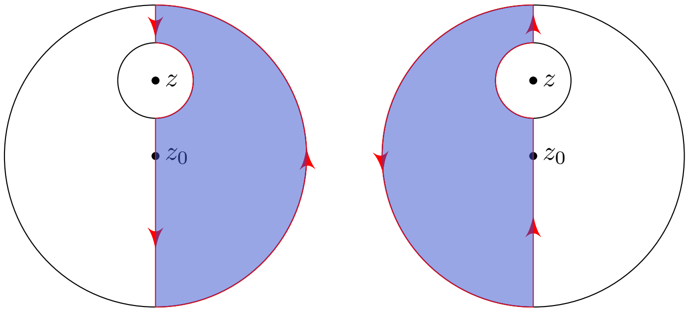
$$|f(z)| = \left|\frac{1}{2\pi i}\int_{\partial\overline{B(z;\rho)}} \frac{f(w)}{w-z} \;\mathrm{d}w\right|$$

Setting $w = z + \rho e^{2\pi i\theta}$, we get

$$\begin{aligned} &= \left|\int_0^1 f(z+\rho e^{2\pi i\theta}) \;\mathrm{d}\theta\right| \\ &\leq \sup_{|z-w|=\rho} |f(w)| \\ &\leq f(z). \end{aligned}$$

So we must have equality throughout. When we proved the supremum bound for the integral, we showed equality can happen only if the integrand is constant. So $|f(w)|$ is constant on the circle $|z-w| = \rho$, and is equal to $f(z)$. Since this is true for all $\rho \in (0,r)$, it follows that $|f|$ is constant on $B(z;r)$. Then the Cauchy–Riemann equations then entail that f must be constant, as you have shown in example sheet 1. □

Proof. (of Cauchy integral formula again) Given $\varepsilon > 0$, we pick $\delta > 0$ such that $\overline{B(z,\delta)} \subseteq B(z_0,r)$, and such that whenever $|w-z| < \delta$, then $|f(w)-f(z)| < \varepsilon$. This is possible since f is uniformly continuous on the neighbourhood of z. We now cut our region apart:



We know $\frac{f(w)}{w-z}$ is holomorphic on sufficiently small open neighbourhoods of the half-contours indicated. The area enclosed by the contours might not be star-shaped, but we can definitely divide it once more so that it is. Hence the integral of $\frac{f(w)}{w-z}$ around the half-contour vanishes by Cauchy's theorem. Adding these together, we get

$$\int_{\partial \overline{B(z_0,r)}} \frac{f(w)}{w-z}\,\mathrm{d}w = \int_{\partial \overline{B(z,\delta)}} \frac{f(w)}{w-z}\,\mathrm{d}w,$$

where the balls are both oriented anticlockwise. Now we have

$$\left|f(z) - \frac{1}{2\pi i}\int_{\partial \overline{B(z_0,r)}} \frac{f(w)}{w-z}\,\mathrm{d}w\right| = \left|f(z) - \frac{1}{2\pi i}\int_{\partial \overline{B(z,\delta)}} \frac{f(w)}{w-z}\,\mathrm{d}w\right|.$$

Now we once again use the fact that

$$\int_{\partial \overline{B(z,\delta)}} \frac{1}{w-z}\,\mathrm{d}z = 2\pi i$$

to show this is equal to

$$\left|\frac{1}{2\pi i}\int_{\partial \overline{B(z,\delta)}} \frac{f(z)-f(w)}{w-z}\,\mathrm{d}w\right| \leq \frac{1}{2\pi}\cdot 2\pi\delta \cdot \frac{1}{\delta}\cdot \varepsilon = \varepsilon.$$

Taking $\varepsilon \to 0$, we see that the Cauchy integral formula holds. $\square$

Theorem (Liouville's theorem)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function (i.e. holomorphic everywhere). If f is bounded, then f is constant.

Proof. Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We fix $z_1, z_2 \in \mathbb{C}$, and estimate $|f(z_1) - f(z_2)|$ with the integral formula.

Let $R > \max\{2|z_1|, 2|z_2|\}$. By the integral formula, we know

$$\begin{aligned}
|f(z_1) - f(z_2)| &= \left|\frac{1}{2\pi i}\int_{\partial B(0,R)} \left(\frac{f(w)}{w-z_1} - \frac{f(w)}{w-z_2}\right)\mathrm{d}w\right|\\
&= \left|\frac{1}{2\pi i}\int_{\partial B(0,R)} \frac{f(w)(z_1-z_2)}{(w-z_1)(w-z_2)}\,\mathrm{d}w\right|\\
&\leq \frac{1}{2\pi}\cdot 2\pi R\cdot \frac{M|z_1-z_2|}{(R/2)^2}\\
&= \frac{4M|z_1-z_2|}{R}.
\end{aligned}$$

Note that we get the bound on the denominator since $|w| = R$ implies $|w - z_i| > \frac{R}{2}$ by our choice of R. Letting $R \to \infty$, we know we must have $f(z_1) = f(z_2)$. So f is constant. $\square$

Corollary (Fundamental theorem of algebra)**.** A non-constant complex polynomial has a root in $\mathbb{C}$.

Proof. Let
$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$
where $a_n \neq 0$ and $n > 0$. So P is non-constant. Thus, as $|z| \to \infty$, $|P(z)| \to \infty$. In particular, there is some R such that for $|z| > R$, we have $|P(z)| \geq 1$.

Now suppose for contradiction that P does not have a root in $\mathbb{C}$. Then consider
$$f(z) = \frac{1}{P(z)},$$
which is then an entire function, since it is a rational function. On $\overline{B(0, R)}$, we know f is certainly continuous, and hence bounded. Outside this ball, we get $|f(z)| \leq 1$. So $f(z)$ is constant, by Liouville's theorem. But P is non-constant. This is absurd. Hence the result follows. $\square$

2.4 Taylor's theorem

Theorem (Taylor's theorem)**.** Let $f : B(a, r) \to \mathbb{C}$ be holomorphic. Then f has a convergent power series representation
$$f(z) = \sum_{n=0}^\infty c_n (z - a)^n$$
on $B(a, r)$. Moreover,
$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\partial B(a, \rho)} \frac{f(z)}{(z - a)^{n + 1}}\;\mathrm{d}z$$
for any $0 < \rho < r$.

Proof. We'll use Cauchy's integral formula. If $|w - a| < \rho < r$, then
$$f(w) = \frac{1}{2\pi i} \int_{\partial B(a, \rho)} \frac{f(z)}{z - w}\;\mathrm{d}z.$$
Now (cf. the first proof of the Cauchy integral formula), we note that
$$\frac{1}{z - w} = \frac{1}{(z - a)\left(1 - \frac{w - a}{z - a}\right)} = \sum_{n=0}^n \frac{(w - a)^n}{(z - a)^{n + 1}}.$$
This series is uniformly convergent everywhere on the ρ disk, including its boundary. By uniform convergence, we can exchange integration and summation to get
$$\begin{aligned}
f(w) &= \sum_{n=0}^\infty \left(\frac{1}{2\pi i} \int_{\partial B(a, \rho)} \frac{f(z)}{(z - a)^{n + 1}}\;\mathrm{d}z\right)(w - a)^n\\
&= \sum_{n=0}^\infty c_n (w - a)^n.
\end{aligned}$$

Since c_n does not depend on w, this is a genuine power series representation, and this is valid on any disk $B(a, \rho) \subseteq B(a, r)$.

Then the formula for c_n in terms of the derivative comes for free since that's the formula for the derivative of a power series. □

Corollary. If $f : B(a, r) \to \mathbb{C}$ is holomorphic on a disc, then f is infinitely differentiable on the disc.

Proof. Complex power series are infinitely differentiable (and f had better be infinitely differentiable for us to write down the formula for c_n in terms of $f^{(n)}$). □

Corollary. If $f : U \to \mathbb{C}$ is a complex-valued function, then $f = u + iv$ is holomorphic at $p \in U$ if and only if u, v satisfy the Cauchy–Riemann equations, and that u_x, u_y, v_x, v_y are continuous in a neighbourhood of p.

Proof. If u_x, u_y, v_x, v_y exist and are continuous in an open neighbourhood of p, then u and v are differentiable as functions $\mathbb{R}^2 \to \mathbb{R}^2$ at p, and then we proved that the Cauchy–Riemann equations imply differentiability at each point in the neighbourhood of p. So f is differentiable at a neighbourhood of p.

On the other hand, if f is holomorphic, then it is infinitely differentiable. In particular, $f'(z)$ is also holomorphic. So u_x, u_y, v_x, v_y are differentiable, hence continuous. □

Corollary (Morera's theorem)**.** Let $U \subseteq \mathbb{C}$ be a domain. Let $f : U \to \mathbb{C}$ be continuous such that

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for all piecewise-C^1 closed curves $\gamma \in U$. Then f is holomorphic on U.

Proof. We have previously shown that the condition implies that f has an antiderivative $F : U \to \mathbb{C}$, i.e. F is a holomorphic function such that $F' = f$. But F is infinitely differentiable. So f must be holomorphic. □

Corollary. Let $U \subseteq \mathbb{C}$ be a domain, $f_n; U \to \mathbb{C}$ be a holomorphic function. If $f_n \to f$ uniformly, then f is in fact holomorphic, and

$$f'(z) = \lim_n f_n'(z).$$

Proof. Given a piecewise C^1 path γ, uniformity of convergence says

$$\int_\gamma f_n(z)\,\mathrm{d}z \to \int_\gamma f(z)\,\mathrm{d}z$$

uniformly. Since f being holomorphic is a local condition, so we fix $p \in U$ and work in some small, convex disc $B(p, \varepsilon) \subseteq U$. Then for any curve γ inside this disk, we have

$$\int_\gamma f_n(z)\,\mathrm{d}z = 0.$$

Hence we also have $\int_\gamma f(z)\,\mathrm{d}z = 0$. Since this is true for all curves, we conclude f is holomorphic inside $B(p, \varepsilon)$ by Morera's theorem. Since p was arbitrary, we know f is holomorphic.

We know the derivative of the limit is the limit of the derivative since we can express $f'(a)$ in terms of the integral of $\frac{f(z)}{(z-a)^2}$, as in Taylor's theorem. □

2.5 Zeroes

Lemma (Principle of isolated zeroes)**.** Let $f : B(a, r) \to \mathbb{C}$ be holomorphic and not identically zero. Then there exists some $0 < \rho < r$ such that $f(z) \neq 0$ in the punctured neighbourhood $B(a, \rho) \setminus \{a\}$.

Proof. If $f(a) \neq 0$, then the result is obvious by continuity of f.

The other option is not too different. If f has a zero of order N at a, then we can write $f(z) = (z - a)^N g(z)$ with $g(a) \neq 0$. By continuity of g, g does not vanish on some small neighbourhood of a, say $B(a, \rho)$. Then $f(z)$ does not vanish on $B(a, \rho) \setminus \{a\}$. □

Corollary (Identity theorem)**.** Let $U \subseteq \mathbb{C}$ be a domain, and $f, g : U \to \mathbb{C}$ be holomorphic. Let $S = \{z \in U : f(z) = g(z)\}$. Suppose S contains a non-isolated point, i.e. there exists some $w \in S$ such that for all $\varepsilon > 0$, $S \cap B(w, \varepsilon) \neq \{w\}$. Then $f = g$ on U.

Proof. Consider the function $h(z) = f(z) - g(z)$. Then the hypothesis says $h(z)$ has a non-isolated zero at w, i.e. there is no non-punctured neighbourhood of w on which h is non-zero. By the previous lemma, this means there is some $\rho > 0$ such that $h = 0$ on $B(w, \rho) \subseteq U$.

Now we do some topological trickery. We let

$$
\begin{aligned}
U_0 &= \{a \in U : h = 0 \text{ on some neighbourhood } B(a, \rho) \text{ of } a \text{ in } U\},\\
U_1 &= \{a \in U : \text{there exists } n \geq 0 \text{ such that } h^{(n)} \neq 0\}.
\end{aligned}
$$

Clearly, $U_0 \cap U_1 = \emptyset$, and the existence of Taylor expansions shows $U_0 \cup U_1 = U$.

Moreover, U_0 is open by definition, and U_1 is open since $h^{(n)}(z)$ is continuous near any given $a \in U_1$. Since U is (path) connected, such a decomposition can happen if one of U_0 and U_1 is empty. But $w \in U_0$. So in fact $U_0 = U$, i.e. h vanishes on the whole of U. So $f = g$. □

2.6 Singularities

Proposition (Removal of singularities)**.** Let U be a domain and $z_0 \in U$. If $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, and f is bounded near z_0, then there exists an a such that $f(z) \to a$ as $z \to z_0$.

Furthermore, if we define

$$
g(z) = \begin{cases} f(z) & z \in U \setminus \{z_0\} \\ a & z = z_0 \end{cases},
$$

then g is holomorphic on U.

Proof. Define a new function $h : U \to \mathbb{C}$ by

$$
h(z) = \begin{cases} (z - z_0)^2 f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}.
$$

Then since f is holomorphic away from z_0, we know h is also holomorphic away from z_0.

Also, we know f is bounded near z_0. So suppose $|f(z)| < M$ in some neighbourhood of z_0. Then we have

$$\left|\frac{h(z) - h(z_0)}{z - z_0}\right| \leq |z - z_0|M.$$

So in fact h is also differentiable at z_0, and $h(z_0) = h'(z_0) = 0$. So near z_0, h has a Taylor series

$$h(z) = \sum_{n \geq 0} a_n (z - z_0)^n.$$

Since we are told that $a_0 = a_1 = 0$, we can define a $g(z)$ by

$$g(z) = \sum_{n \geq 0} a_{n+2} (z - z_0)^n,$$

defined on some ball $B(z_0, \rho)$, where the Taylor series for h is defined. By construction, on the punctured ball $B(z_0, \rho) \setminus \{z_0\}$, we get $g(z) = f(z)$. Moreover, $g(z) \to a_2$ as $z \to z_0$. So $f(z) \to a_2$ as $z \to z_0$.

Since g is a power series, it is holomorphic. So the result follows. □

Proposition. Let U be a domain, $z_0 \in U$ and $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Suppose $|f(z)| \to \infty$ as $z \to z_0$. Then there is a unique $k \in \mathbb{Z}_{\geq 1}$ and a unique holomorphic function $g : U \to \mathbb{C}$ such that $g(z_0) \neq 0$, and

$$f(z) = \frac{g(z)}{(z - z_0)^k}.$$

Proof. We shall construct g near z_0 in some small neighbourhood, and then apply analytic continuation to the whole of U. The idea is that since $f(z)$ blows up nicely as $z \to z_0$, we know $\frac{1}{f(z)}$ behaves sensibly near z_0.

We pick some $\delta > 0$ such that $|f(z)| \geq 1$ for all $z \in B(z_0; \delta) \setminus \{z_0\}$. In particular, $f(z)$ is non-zero on $B(z_0; \delta) \setminus \{z_0\}$. So we can define

$$h(z) = \begin{cases} \frac{1}{f(z)} & z \in B(z_0; \delta) \setminus \{z_0\} \\ 0 & z = z_0 \end{cases}.$$

Since $|\frac{1}{f(z)}| \leq 1$ on $B(z_0; \delta) \setminus \{z_0\}$, by the removal of singularities, h is holomorphic on $B(z_0, \delta)$. Since h vanishes at the z_0, it has a unique definite order at z_0, i.e. there is a unique integer $k \geq 1$ such that h has a zero of order k at z_0. In other words,

$$h(z) = (z - z_0)^k \ell(z),$$

for some holomorphic $\ell : B(z_0; \delta) \to \mathbb{C}$ and $\ell(z_0) \neq 0$.

Now by continuity of ℓ, there is some $0 < \varepsilon < \delta$ such that $\ell(z) \neq 0$ for all $z \in B(z_0, \varepsilon)$. Now define $g : B(z_0; \varepsilon) \to \mathbb{C}$ by

$$g(z) = \frac{1}{\ell(z)}.$$

Then g is holomorphic on this disc.

By construction, at least away from z_0, we have

$$g(z) = \frac{1}{\ell(z)} = \frac{1}{h(z)} \cdot (z - z_0)^k = (z - z_0)^k f(z).$$

g was initially defined on $B(z_0; \varepsilon) \to \mathbb{C}$, but now this expression certainly makes sense on all of U. So g admits an analytic continuation from $B(z_0; \varepsilon)$ to U. So done. $\square$

Theorem (Casorati-Weierstrass theorem)**.** Let U be a domain, $z_0 \in U$, and suppose $f : U \setminus \{z_0\} \to \mathbb{C}$ has an essential singularity at z_0. Then for all $w \in \mathbb{C}$, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w$.

In other words, on any punctured neighbourhood $B(z_0; \varepsilon) \setminus \{z_0\}$, the image of f is dense in $\mathbb{C}$.

Proof. See example sheet 2. $\square$

Theorem (Picard's theorem)**.** If f has an isolated essential singularity at z_0, then there is some $b \in \mathbb{C}$ such that on each punctured neighbourhood $B(z_0; \varepsilon) \setminus \{z_0\}$, the image of f contains $\mathbb{C} \setminus \{b\}$.

2.7 Laurent series

Theorem (Laurent series)**.** Let $0 \leq r < R < \infty$, and let

$$A = \{z \in \mathbb{C} : r < |z - a| < R\}$$

denote an annulus on $\mathbb{C}$.

Suppose $f : A \to \mathbb{C}$ is holomorphic. Then f has a (unique) convergent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n,$$

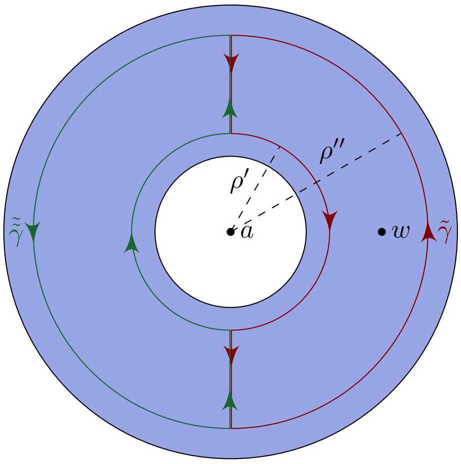
where

$$c_n = \frac{1}{2\pi i} \int_{\partial \overline{B(a, \rho)}} \frac{f(z)}{(z - a)^{n+1}} \, \mathrm{d}z$$

for $r < \rho < R$. Moreover, the series converges uniformly on compact subsets of the annulus.

Proof. The proof looks very much like the blend of the two proofs we've given for the Cauchy integral formula. In one of them, we took a power series expansion of the integrand, and in the second, we changed our contour by cutting it up. This is like a mix of the two.

Let $w \in A$. We let $r < \rho' < |w - a| < \rho'' < R$.



We let $\tilde{\gamma}$ be the contour containing w, and $\tilde{\tilde{\gamma}}$ be the other contour.

Now we apply the Cauchy integral formula to say

$$f(w) = \frac{1}{2\pi i}\int_{\tilde{\gamma}} \frac{f(z)}{z - w}\;\mathrm{d}z$$

and

$$0 = \frac{1}{2\pi i}\int_{\tilde{\tilde{\gamma}}} \frac{f(z)}{z - w}\;\mathrm{d}z.$$

So we get

$$f(w) = \frac{1}{2\pi i}\int_{\partial B(a, \rho'')}\frac{f(z)}{z - w}\;\mathrm{d}z - \frac{1}{2\pi i}\int_{\partial B(a, \rho')}\frac{f(z)}{z - w}\;\mathrm{d}z.$$

As in the first proof of the Cauchy integral formula, we make the following expansions: for the first integral, we have $w - a < z - a$. So

$$\frac{1}{z - w} = \frac{1}{z - a}\left(\frac{1}{1 - \frac{w - a}{z - a}}\right) = \sum_{n = 0}^\infty \frac{(w - a)^n}{(z - a)^{n + 1}},$$

which is uniformly convergent on $z \in \partial B(a, \rho'')$.

For the second integral, we have $w - a > z - a$. So

$$\frac{-1}{z - w} = \frac{1}{w - a}\left(\frac{1}{1 - \frac{z - a}{w - a}}\right) = \sum_{m = 1}^\infty \frac{(z - a)^{m - 1}}{(w - a)^m},$$

which is uniformly convergent for $z \in \partial B(a, \rho')$.

By uniform convergence, we can swap summation and integration. So we get

$$\begin{aligned}f(w) = \sum_{n = 0}^\infty &\left(\frac{1}{2\pi i}\int_{\partial B(a, \rho'')}\frac{f(z)}{(z - a)^{n + 1}}\;\mathrm{d}z\right) (w - a)^n\\ &+ \sum_{m = 1}^\infty\left(\frac{1}{2\pi i}\int_{\partial B(a, \rho')} \frac{f(z)}{(z - a)^{-m + 1}}\;\mathrm{d}z\right)(w - a)^{-m}.\end{aligned}$$

Now we substitute $n = -m$ in the second sum, and get

$$f(w) = \sum_{n=-\infty}^{\infty} \tilde{c}_n (w-a)^n,$$

for the integrals $\tilde{c}_n$. However, some of the coefficients are integrals around the ρ'' circle, while the others are around the ρ' circle. This is not a problem. For any $r < \rho < R$, these circles are convex deformations of $|z-a| = \rho$ inside the annulus A. So

$$\int_{\partial B(a,\rho)} \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z$$

is independent of ρ as long as $\rho \in (r, R)$. So we get the result stated. □

Lemma. Let $f : A \to \mathbb{C}$ be holomorphic, $A = \{r < |z-a| < R\}$, with

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

Then the coefficients c_n are uniquely determined by f.

Proof. Suppose also that

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a)^n.$$

Using our formula for c_k, we know

$$\begin{aligned}
2\pi i c_k &= \int_{\partial B(a,\rho)} \frac{f(z)}{(z-a)^{k+1}}\,\mathrm{d}z \\
&= \int_{\partial B(a,\rho)} \left(\sum_n b_n (z-a)^{n-k-1}\right)\,\mathrm{d}z \\
&= \sum_n b_n \int_{\partial B(a,\rho)} (z-a)^{n-k-1}\,\mathrm{d}z \\
&= 2\pi i b_k.
\end{aligned}$$

So $c_k = b_k$. □

3 Residue calculus

3.1 Winding numbers

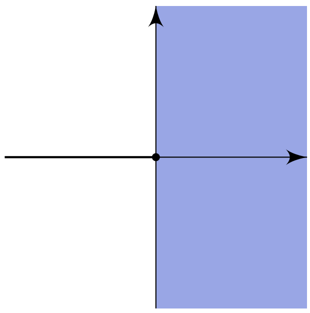
Lemma. Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous closed curve, and pick a point $w \in \mathbb{C} \setminus \operatorname{image}(\gamma)$. Then there are continuous functions $r : [a, b] \to \mathbb{R} > 0$ and $\theta : [a, b] \to \mathbb{R}$ such that

$$\gamma(t) = w + r(t)e^{i\theta(t)}.$$

Proof. Clearly $r(t) = |\gamma(t) - w|$ exists and is continuous, since it is the composition of continuous functions. Note that this is never zero since $\gamma(t)$ is never w. The actual content is in defining θ.

To define $\theta(t)$, we for simplicity assume $w = 0$. Furthermore, by considering instead the function $\frac{\gamma(t)}{r(t)}$, which is continuous and well-defined since r is never zero, we can assume $|\gamma(t)| = 1$ for all t.

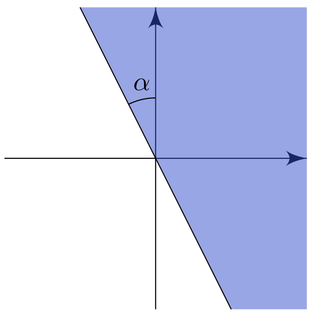
Recall that the principal branch of log, and hence of the argument Im(log), takes values in $(-\pi, \pi)$ and is defined on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.



If $\gamma(t)$ always lied in, say, the right-hand half plane, we would have no problem defining θ consistently, since we can just let

$$\theta(t) = \arg(\gamma(t))$$

for arg the principal branch. There is nothing special about the right-hand half plane. Similarly, if γ lies in the region as shaded below:



i.e. we have

$$\gamma(t) \in \left\{z : \operatorname{Re}\left(\frac{z}{e^{i\alpha}}\right) > 0\right\}$$

for a fixed α, we can define

$$\theta(t) = \alpha + \arg\left(\frac{\gamma(t)}{e^{i\alpha}}\right).$$

Since $\gamma : [a, b] \to \mathbb{C}$ is continuous, it is uniformly continuous, and we can find a subdivision
$$a = a_0 < a_1 < \cdots < a_m = b,$$
such that if $s, t \in [a_{i-1}, a_i]$, then $|\gamma(s) - \gamma(t)| < \sqrt{2}$, and hence $\gamma(s)$ and $\gamma(t)$ belong to such a half-plane.

So we define $\theta_j : [a_{j-1}, a_j] \to \mathbb{R}$ such that
$$\gamma(t) = e^{i\theta_j(t)}$$
for $t \in [a_{j-1}, a_j]$, and $1 \leq j \leq n - 1$.

On each region $[a_{j-1}, a_j]$, this gives a continuous argument function. We cannot immediately extend this to the whole of $[a, b]$, since it is entirely possible that $\theta_j(a_j) = \theta_{j+1}(a_j)$. However, we do know that $\theta_j(a_j)$ are both values of the argument of $\gamma(a_j)$. So they must differ by an integer multiple of 2π, say $2n\pi$. Then we can just replace θ_{j+1} by $\theta_{j+1} - 2n\pi$, which is an equally valid argument function, and then the two functions will agree at a_j.

Hence, for $j > 1$, we can successively re-define θ_j such that the resulting map θ is continuous. Then we are done. □

Lemma. Suppose $\gamma : [a, b] \to \mathbb{C}$ is a piecewise C^1-smooth closed path, and $w \notin \text{image}(\gamma)$. Then
$$I(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - w}\,\mathrm{d}z.$$

Proof. Let $\gamma(t) - w = r(t)e^{i\theta(t)}$, with now r and θ piecewise C^1-smooth. Then
$$\begin{aligned}\int_\gamma \frac{1}{z - w}\,\mathrm{d}z &= \int_a^b \frac{\gamma'(t)}{\gamma(t) - w}\,\mathrm{d}t\\ &= \int_a^b \left(\frac{r'(t)}{r(t)} + i\theta'(t)\right)\,\mathrm{d}t\\ &= [\ln r(t) + i\theta(t)]_a^b\\ &= i(\theta(b) - \theta(a))\\ &= 2\pi i I(\gamma, w).\end{aligned}$$
So done. □

3.2 Homotopy of closed curves

Proposition. Let $\phi, \psi : [a, b] \to U$ be homotopic (piecewise C^1) closed paths in a domain U. Then there exists some $\phi = \phi_0, \phi_1, \cdots, \phi_N = \psi$ such that each ϕ_j is piecewise C^1 closed and ϕ_{i+1} is obtained from ϕ_i by elementary deformation.

Proof. This is an exercise in uniform continuity. We let $F : [0, 1] \times [a, b] \to U$ be a homotopy from ϕ to ψ. Since image(F) is compact and U is open, there is some $\varepsilon > 0$ such that $B(F(s, t), \varepsilon) \subseteq U$ for all $(s, t) \in [0, 1] \times [a, b]$ (for each s, t, pick the maximum $\varepsilon_{s,t} > 0$ such that $B(F(s, t), \varepsilon_{s,t}) \subseteq U$. Then $\varepsilon_{s,t}$ varies continuously with s, t, hence attains its minimum on the compact set $[0, 1] \times [a, b]$. Then picking ε to be the minimum works).

Since F is uniformly continuous, there is some δ such that $\|(s, t) - (s', t')\| < \delta$ implies $|F(s, t) - F(s', t')| < \varepsilon$.

Now we pick $n \in \mathbb{N}$ such that $\frac{1+(b-a)}{n} < \delta$, and let

$$\begin{aligned} x_j &= a + (b-a)\frac{j}{n} \\ \phi_i(t) &= F\left(\tfrac{i}{n}, t\right) \\ C_{ij} &= B\left(F\left(\tfrac{i}{n}, x_j\right), \varepsilon\right) \end{aligned}$$

Then C_{ij} is clearly convex. These definitions are cooked up precisely so that if $s \in \left(\frac{i-1}{n}, \frac{i}{n}\right)$ and $t \in [x_{j-1}, x_j]$, then $F(s,t) \in C_{ij}$. So the result follows. $\square$

Corollary. Let U be a domain, $f : U \to \mathbb{C}$ be holomorphic, and γ_1, γ_2 be homotopic piecewise C^1-smooth closed curves in U. Then

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

Corollary (Cauchy's theorem for simply connected domains)**.** Let U be a simply connected domain, and let $f : U \to \mathbb{C}$ be holomorphic. If γ is any piecewise C^1-smooth closed curve in U, then

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. By definition of simply-connected, γ is homotopic to the constant path, and it is easy to see the integral along a constant path is zero. $\square$

3.3 Cauchy's residue theorem

Theorem (Cauchy's residue theorem)**.** Let U be a simply connected domain, and $\{z_1, \cdots, z_k\} \subseteq U$. Let $f : U \setminus \{z_1, \cdots, z_k\} \to \mathbb{C}$ be holomorphic. Let $\gamma : [a,b] \to U$ be a piecewise C^1-smooth closed curve such that $z_i \neq \mathrm{image}(\gamma)$ for all i. Then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^k I(\gamma, z_i)\operatorname{Res}(f; z_i).$$

Proof. At each z_i, f has a Laurent expansion

$$f(z) = \sum_{n\in\mathbb{Z}} c_n^{(i)}(z - z_i)^n,$$

valid in some neighbourhood of z_i. Let $g_i(z)$ be the principal part, namely

$$g_i(z) = \sum_{n=-\infty}^{-1} c_n^{(i)}(z - z_i)^n.$$

From the proof of the Laurent series, we know $g_i(z)$ gives a holomorphic function on $U \setminus \{z_i\}$.

We now consider $f - g_1 - g_2 - \cdots - g_k$, which is holomorphic on $U \setminus \{z_1, \cdots, z_k\}$, and has a *removable* singularity at each z_i. So

$$\int_\gamma (f - g_1 - \cdots - g_k)(z)\,\mathrm{d}z = 0,$$

by simply-connected Cauchy. Hence we know

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^k \int_\gamma g_j(z)\,\mathrm{d}z.$$

For each j, we use uniform convergence of the series $\sum_{n\leq -1} c_n^{(j)}(z-z_j)^n$ on compact subsets of $U \setminus \{z_j\}$, and hence on γ, to write

$$\int_\gamma g_j(z)\,\mathrm{d}z = \sum_{n\leq -1} c_n^{(j)} \int_\gamma (z-z_j)^n\,\mathrm{d}z.$$

However, for $n \neq -1$, the function $(z-z_j)^n$ has an antiderivative, and hence the integral around γ vanishes. So this is equal to

$$c_{-1}^{(j)} \int_\gamma \frac{1}{z-z_j}\,\mathrm{d}z.$$

But $c_{-1}^{(j)}$ is by definition the residue of f at z_j, and the integral is just the integral definition of the winding number (up to a factor of $2\pi i$). So we get

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^k \operatorname{Res}(f; z_j) I(\gamma, z_j).$$

So done. □

3.4 Overview

3.5 Applications of the residue theorem

Lemma. Let $f : U \setminus \{a\} \to \mathbb{C}$ be holomorphic with a pole at a, i.e f is meromorphic on U.

(i) If the pole is simple, then

$$\operatorname{Res}(f, a) = \lim_{z\to a}(z-a)f(z).$$

(ii) If near a, we can write

$$f(z) = \frac{g(z)}{h(z)},$$

where $g(a) \neq 0$ and h has a simple zero at a, and g, h are holomorphic on $B(a, \varepsilon) \setminus \{a\}$, then

$$\operatorname{Res}(f, a) = \frac{g(a)}{h'(a)}.$$

(iii) If

$$f(z) = \frac{g(z)}{(z-a)^k}$$

near a, with $g(a) \neq 0$ and g is holomorphic, then

$$\operatorname{Res}(f, a) = \frac{g^{(k-1)}(a)}{(k-1)!}.$$

Proof.

(i) By definition, if f has a simple pole at a, then

$$f(z) = \frac{c_{-1}}{(z-a)} + c_0 + c_1(z-a) + \cdots,$$

and by definition $c_{-1} = \mathrm{Res}(f, a)$. Then the result is obvious.

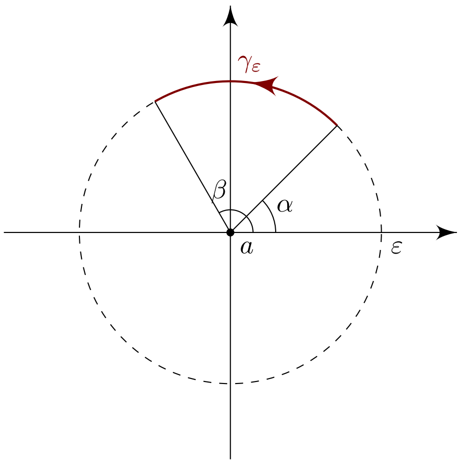
(ii) This is basically L'Hôpital's rule. By the previous part, we have

$$\mathrm{Res}(f; a) = \lim_{z \to a} (z-a)\frac{g(z)}{h(z)} = g(a) \lim_{z \to a} \frac{z-a}{h(z) - h(a)} = \frac{g(a)}{h'(a)}.$$

(iii) We know the residue $\mathrm{Res}(f; a)$ is the coefficient of $(z-a)^{k-1}$ in the Taylor series of g at a, which is exactly $\frac{1}{(k-1)!} g^{(k-1)}(a)$. □

Lemma. Let $f : B(a, r) \setminus \{a\} \to \mathbb{C}$ be holomorphic, and suppose f has a simple pole at a. We let $\gamma_\varepsilon : [\alpha, \beta] \to \mathbb{C}$ be given by

$$t \mapsto a + \varepsilon e^{it}.$$



Then

$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = (\beta - \alpha) \cdot i \cdot \mathrm{Res}(f, a).$$

Proof. We can write

$$f(z) = \frac{c}{z-a} + g(z)$$

near a, where $c = \mathrm{Res}(f; a)$, and $g : B(a, \delta) \to \mathbb{C}$ is holomorphic near a. We take $\varepsilon < \delta$. Then

$$\left| \int_{\gamma_\varepsilon} g(z)\,\mathrm{d}z \right| \leq (\beta - \alpha) \cdot \varepsilon \sup_{z \in \gamma_\varepsilon} |g(z)|.$$

But g is bounded on $B(\alpha, \delta)$. So this vanishes as $\varepsilon \to 0$. So the remaining integral is

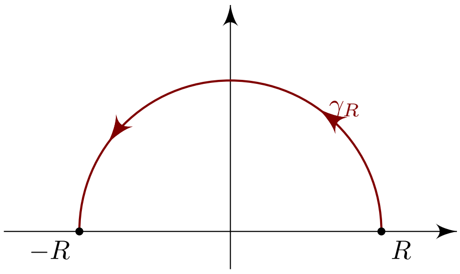
$$\begin{aligned}\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \frac{c}{z-a}\,\mathrm{d}z &= c \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\frac{1}{z-a}\,\mathrm{d}z\\ &= c\lim_{\varepsilon\to 0}\int_\alpha^\beta \frac{1}{\varepsilon e^{it}}\cdot i\varepsilon e^{it}\,\mathrm{d}t\\ &= i(\beta-\alpha)c,\end{aligned}$$

as required. □

Lemma (Jordan's lemma)**.** Let f be holomorphic on a neighbourhood of infinity in $\mathbb{C}$, i.e. on $\{|z| > r\}$ for some $r > 0$. Assume that $zf(z)$ is bounded in this region. Then for $\alpha > 0$, we have

$$\int_{\gamma_R} f(z)e^{i\alpha z}\,\mathrm{d}z \to 0$$

as $R \to \infty$, where $\gamma_R(t) = Re^{it}$ for $t \in [0, \pi]$ is the semicircle (which is *not* closed).



Proof. By assumption, we have

$$|f(z)| \leq \frac{M}{|z|}$$

for large $|z|$ and some constant $M > 0$. We also have

$$|e^{i\alpha z}| = e^{-R\alpha\sin t}$$

on γ_R. To avoid messing with $\sin t$, we note that on $(0, \frac{\pi}{2}]$, the function $\frac{\sin\theta}{\theta}$ is decreasing, since

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\theta}\right) = \frac{\theta\cos\theta - \sin\theta}{\theta^2} \leq 0.$$

Then by consider the end points, we find

$$\sin(t) \geq \frac{2t}{\pi}$$

for $t \in [0, \frac{\pi}{2}]$. This gives us the bound

$$|e^{i\alpha z}| = e^{-R\alpha\sin t} \leq \begin{cases} e^{-Ra2t/\pi} & 0 \leq t \leq \frac{\pi}{2}\\ e^{-Ra2t'/\pi} & 0 \leq t' = \pi - t \leq \frac{\pi}{2}\end{cases}$$

So we get

$$\left|\int_0^{\pi/2} e^{iR\alpha e^{it}} f(Re^{it}) Re^{it}\ \mathrm{d}t\right| \leq \int_0^{2\pi} e^{-2\alpha Rt/\pi} \cdot M\ \mathrm{d}t$$

$$= \frac{1}{2R}(1 - e^{\alpha R})$$

$$\to 0$$

as $R \to \infty$.

The estimate for

$$\int_{\pi/2}^{\pi} f(z) e^{i\alpha z}\ \mathrm{d}z$$

is analogous. □

3.6 Rouchés theorem

Theorem (Argument principle)**.** Let U be a simply connected domain, and let f be meromorphic on U. Suppose in fact f has finitely many zeroes $z_1, \cdots, z_k$ and finitely many poles $w_1, \cdots, w_\ell$. Let γ be a piecewise-C^1 closed curve such that $z_i, w_j \notin \mathrm{image}(\gamma)$ for all i, j. Then

$$I(f \circ \gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\ \mathrm{d}z = \sum_{i=1}^{k} \mathrm{ord}(f; z_i) I_\gamma(z_i) - \sum_{j=1}^{\ell} \mathrm{ord}(f, w_j) I(\gamma, w_j).$$

Proof. By the residue theorem, we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\ \mathrm{d}z = \sum_{z \in U} \mathrm{Res}\left(\frac{f'}{f}, z\right) I(\gamma, z),$$

where we sum over all zeroes and poles of z. Note that outside these zeroes and poles, the function $\frac{f'(z)}{f(z)}$ is holomorphic.

Now at each z_i, if $f(z) = (z - z_j)^k g(z)$, with $g(z_j) \neq 0$, then by direct computation, we get

$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_j} + \frac{g'(z)}{g(z)}.$$

Since at z_j, g is holomorphic and non-zero, we know $\frac{g'(z)}{g(z)}$ is holomorphic near z_j. So

$$\mathrm{Res}\left(\frac{f'}{f}, z_j\right) = k = \mathrm{ord}(f, z_j).$$

Analogously, by the same proof, at the w_i, we get

$$\mathrm{Res}\left(\frac{f'}{f}, w_j\right) = -\,\mathrm{ord}(f; w_j).$$

So done. □

Corollary (Rouchés theorem)**.** Let U be a domain and γ a closed curve which bounds a domain in U (the key case is when U is simply connected and γ is a simple closed curve). Let f, g be holomorphic on U, and suppose $|f| > |g|$ for all $z \in \mathrm{image}(\gamma)$. Then f and $f + g$ have the same number of zeroes in the domain bound by γ, when counted with multiplicity.

Proof. If $|f| > |g|$ on γ, then f and $f+g$ cannot have zeroes on the curve γ. We let
$$h(z) = \frac{f(z)+g(z)}{f(z)} = 1 + \frac{g(z)}{f(z)}.$$
This is a natural thing to consider, since zeroes of $f+g$ is zeroes of h, while poles of h are zeroes of f. Note that by assumption, for all $z \in \gamma$, we have
$$h(z) \in B(1,1) \subseteq \{z : \operatorname{Re} z > 0\}.$$
Therefore $h \circ \gamma$ is a closed curve in the half-plane $\{z : \operatorname{Re} z > 0\}$. So $I(h\circ\gamma; 0) = 0$. Then by the argument principle, h must have the same number of zeros as poles in D, when counted with multiplicity (note that the winding numbers are all $+1$).

Thus, as the zeroes of h are the zeroes of $f+g$, and the poles of h are the poles of f, the result follows. □

Lemma. The local degree is given by
$$\deg(f,a) = I(f\circ\gamma, f(a)),$$
where
$$\gamma(t) = a + re^{it},$$
with $0 \leq t \leq 2\pi$, for $r > 0$ sufficiently small.

Proof. Note that by the identity theorem, we know that, $f(z) - f(a)$ has an isolated zero at a (since f is non-constant). So for sufficiently small r, the function $f(z) - f(a)$ does not vanish on $\overline{B(a,r)} \setminus \{a\}$. If we use this r, then $f\circ\gamma$ never hits $f(a)$, and the winding number is well-defined.

The result then follows directly from the argument principle. □

Proposition (Local degree theorem)**.** Let $f : B(a,r) \to \mathbb{C}$ be holomorphic and non-constant. Then for $r > 0$ sufficiently small, there is $\varepsilon > 0$ such that for any $w \in B(f(a),\varepsilon) \setminus \{f(a)\}$, the equation $f(z) = w$ has exactly $\deg(f,a)$ distinct solutions in $B(a,r)$.

Proof. We pick $r > 0$ such that $f(z) - f(a)$ and $f'(z)$ don't vanish on $B(a,r)\setminus\{a\}$. We let $\gamma(t) = a + re^{it}$. Then $f(a) \notin \operatorname{image}(f\circ\gamma)$. So there is some $\varepsilon > 0$ such that
$$B(f(a),\varepsilon) \cap \operatorname{image}(f\circ\gamma) = \emptyset.$$
We now let $w \in B(f(a),\varepsilon)$. Then the number of zeros of $f(z) - w$ in $B(a,r)$ is just $I(f\circ\gamma, w)$, by the argument principle. This is just equal to $I(f\circ\gamma, f(a)) = \deg(f,a)$, by the invariance of $I(\Gamma, *)$ as we move $*$ in a component $\mathbb{C}\setminus\Gamma$.

Now if $w \neq f(a)$, since $f'(z) \neq 0$ on $B(a,r)\setminus\{a\}$, all roots of $f(z) - w$ must be simple. So there are exactly $\deg(f;a)$ distinct zeros. □

Corollary (Open mapping theorem)**.** Let U be a domain and $f : U \to \mathbb{C}$ is holomorphic and non-constant, then f is an open map, i.e. for all open $V \subseteq U$, we get that $f(V)$ is open.

Proof. This is an immediate consequence of the local degree theorem. It suffices to prove that for every $z \in U$ and $r > 0$ sufficiently small, we can find $\varepsilon > 0$ such that $B(f(a),\varepsilon) \subseteq f(B(a,r))$. This is true by the local degree theorem. □

Corollary. Let $U \subseteq \mathbb{C}$ be a simply connected domain, and $U \neq \mathbb{C}$. Then there is a non-constant holomorphic function $U \to B(0, 1)$.

Proof. We let $q \in \mathbb{C} \setminus U$, and let $\phi(z) = z - q$. So $\phi : U \to \mathbb{C}$ is non-vanishing. It is also clearly holomorphic and non-constant. By an exercise (possibly on the example sheet), there is a holomorphic function $g : U \to \mathbb{C}$ such that $\phi(z) = e^{g(z)}$ for all z. In particular, our function $\phi(z) = z - q : U \to \mathbb{C}^*$ can be written as $\phi(z) = h(z)^2$, for some function $h : U \to \mathbb{C}^*$ (by letting $h(z) = e^{\frac{1}{2}g(z)}$).

We let $y \in h(U)$, and then the open mapping theorem says there is some $r > 0$ with $B(y, r) \subseteq h(U)$. But notice ϕ is injective by observation, and that $h(z_1) = \pm h(z_2)$ implies $\phi(z_1) = \phi(z_2)$. So we deduce that $B(-y, r) \cap h(U) = \emptyset$ (note that since $y \neq 0$, we have $B(y, r) \cap B(-y, r) = \emptyset$ for sufficiently small r).

Now define
$$f : z \mapsto \frac{r}{2(h(z) + y)}.$$
This is a holomorphic function $f : U \to B(0, 1)$, and is non-constant. $\square$