Complex Analysis IB: 2015-16 – Sheet 1

1. Let $T : \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$ be a real linear map. Show that T can be written $Tz = Az + B\bar{z}$ for unique $A, B \in \mathbb{C}$. Show that T is complex differentiable if and only if $B = 0$.

2. (i) Let $f : D \to \mathbb{C}$ be a holomorphic function defined on a domain (non-empty path-connected open subset) D. Show that f is constant if any of its real part, modulus or argument is constant.

   (ii) Find all holomorphic functions on $\mathbb{C}$ of the form $f(x + iy) = u(x) + iv(y)$ where u and v are both real valued.

   (iii) Find all holomorphic functions $f(z)$ on $\mathbb{C}$ which have real part $x^3 - 3xy^2$.

3. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$, and

$$f(z) = \frac{(1 + i)x^3 - (1 - i)y^3}{x^2 + y^2} \quad \text{for } z = x + iy \neq 0.$$

   Show that f satisfies the Cauchy-Riemann equations at 0 but is not differentiable there.

4. (i) Verify directly that e^z and $\cos z$ satisfy the Cauchy-Riemann equations everywhere.

   (ii) Find the set of $z \in \mathbb{C}$ for which $\left|e^{iz}\right| > 1$, and the set of those for which $|e^z| \leq e^{|z|}$.

   (iii) Find the zeros of $1 + e^z$ and $\cosh z$.

5. (i) Defining $z^\alpha = e^{\alpha \operatorname{Log} z}$, for Log the principal branch of the logarithm and $z \notin \mathbb{R}_{\leq 0}$, show that $\frac{d}{dz} z^\alpha = \alpha\, z^{\alpha - 1}$. Does $(zw)^\alpha = z^\alpha w^\alpha$ always hold ?

   (ii) If $z \in \mathbb{C}$, show that $n \operatorname{Log}(1 + z/n)$ is defined if n is sufficiently large, and that it tends to z as n tends to ∞. Deduce

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z \qquad \forall\, z \in \mathbb{C}.$$

6. Find conformal equivalences between the following pairs of domains:

   (i) the sector $\{z \in \mathbb{C} \mid -\pi/4 < \arg(z) < \pi/4\}$ and the open unit disc D;

   (ii) the lune $\{z \in \mathbb{C} : |z - 1| < \sqrt{2} \text{ and } |z + 1| < \sqrt{2}\}$ and the open unit disk D;

   (iii) the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$ and the quadrant $Q = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$.

   By considering a suitable bounded solution of the Laplace equation $u_{xx} + u_{yy} = 0$ on the strip S, find a non-constant harmonic function on Q which is constant on each of the two boundaries of the quadrant (it need not be continuous at the origin).

7. (i) Show that the general Möbius transformation which takes the unit disk to itself has the form $z \mapsto \lambda \frac{z - a}{\bar{a}z - 1}$, with $|a| < 1$, $|\lambda| = 1$. [*Hint: first show these maps form a group.*]

   (ii) Find a Möbius transformation taking the region between $\{|z| = 1\}$ and $\{|z - 1| = 5/2\}$ to an annulus $\{1 < |z| < R\}$. [*Hint: A circle can be described by an equation of the shape* $|z - a|/|z - b| = l$.]

   (iii) Find a conformal map from an infinite strip onto an annulus. Can such a map be the restriction to the strip of a Möbius map ?

8. Prove that the following series converge uniformly on compact (i.e. closed and bounded) subsets of the given domains in $\mathbb{C}$:

$$(a)\ \sum_{n=1}^{\infty} \sqrt{n} e^{-nz}, \quad \text{on } \{0 < \mathrm{Re}(z)\}; \qquad (b)\ \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}}, \quad \text{on } \left\{|z| < \frac{1}{2}\right\}.$$

9. Calculate $\int_\gamma z \sin z \, dz$ when γ is the straight line joining 0 to i.

10. Show that the following functions do not have antiderivatives (i.e. functions of which they are the derivative) on the domains indicated:

$$(a)\ \frac{1}{z} - \frac{1}{z-1} \quad (0 < |z| < 1); \qquad (b)\ \frac{z}{1+z^2} \quad (1 < |z| < \infty).$$

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Complex Analysis IB: 2015-16 – Sheet 2

Let $B(a;\epsilon)$ denote the open ball $\{z \in \mathbb{C} : |z-a| < \epsilon\}$.

1. The Weierstrass approximation theorem states that any continuous function $f : I \to \mathbb{R}$ on a closed bounded connected subset $I \subset \mathbb{R}$ can be uniformly approximated by polynomials. Can any continuous function $\phi : J \to \mathbb{C}$ on a closed bounded connected subset $J \subset \mathbb{C}$ be uniformly approximated by polynomials? Justify your answer.

2. (i) Using the Cauchy integral formula, compute $\int_{|z|=2} \frac{1}{z^2+1}\, dz$ and $\int_{|z|=2} \frac{1}{z^2-1}\, dz$.

   (ii) If $p(z)$ is a polynomial with distinct roots $\{a_j\}$, what is the maximum conceivable number of distinct values that $\int_\gamma \frac{1}{p(z)}\, dz$ can take, as γ varies over (piecewise C^1-smooth) simple closed curves disjoint from the $\{a_j\}$? [You are not asked to provide a polynomial p for which this theoretical maximum is realised.]

3. (i) For $\alpha \in \mathbb{C}$, use the Cauchy integral formula to compute

$$\int_{|z|=1} \frac{e^{\alpha z}}{2z^2 - 5z + 2}\, dz.$$

   (ii) By considering suitable complex integrals, show that

$$\int_0^\pi \frac{\cos n\theta}{1 - 2r\cos\theta + r^2}\, d\theta = \frac{\pi r^n}{1-r^2} \ \forall\, r \in (0,1); \quad \text{and} \quad \int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta \ = \ 2\pi.$$

4. Let f be an entire function.

   (i) If $f(z)/z \to 0$ as $|z| \to \infty$, prove that f is constant. (This strengthens Liouville's theorem.)

   (ii) If for some $a \in \mathbb{C}$ and $\epsilon > 0$, f never takes values in $B(a;\epsilon)$, show that f is constant.

   (iii) If $f = u + iv$ and $|u| > |v|$ throughout $\mathbb{C}$, show that f is constant.

5. Let U be a domain and $f : U \to \mathbb{C}$ be holomorphic. If the real part $\mathrm{Re}(f)$ has an interior local maximum at $a \in U$, show that f is constant.

6. (i) Let f be an entire function. Show that f is a polynomial, of degree $\le k$, if and only if there is a constant M for which $|f(z)| < M(1+|z|)^k$ for all z.

   (ii) Show that an entire function f is a polynomial if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

   (iii) Let f be a function which is holomorphic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists k such that $|f(z)| \le |z|^k$ for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

7. (i) (Schwarz's Lemma) Let f be holomorphic on the open unit disk $B(0;1)$, satisfying $|f(z)| \le 1$ and $f(0) = 0$. By applying the maximum principle to $f(z)/z$, show that $|f(z)| \le |z|$. Show also that if $|f(w)| = |w|$ for some $w \ne 0$ then $f(z) = cz$ for some constant c.

   (ii) Use Schwarz's Lemma to prove that any conformal equivalence from the unit disk to itself is given by a Möbius transformation.

8. (i) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. If $f(1/n) = 1/n$ for each $n \in \mathbb{Z}_{>0}$, show that $f(z) = z$.

   (ii) Let $f\colon \mathbb{C} \to \mathbb{C}$ be holomorphic. If $f(n) = n^2$ for every $n \in \mathbb{Z}$, must $f(z) = z^2$?

   (iii) Let f be holomorphic on $B(0; 2)$. Show that $f(1/n) \neq 1/(n + 1)$ for some $n \in \mathbb{Z}_{>0}$.

9. (i) Give an example of an infinitely differentiable function $f : (-1, 1) \to \mathbb{R}$ which can be extended to a holomorphic function on a domain $(-1, 1) \subset U \subset \mathbb{C}$, but for which one cannot take U to be the open unit disc $B(0; 1)$.

   (ii) Give an example of an infinitely differentiable function $f : (-1, 1) \to \mathbb{R}$ which is not the restriction of any holomorphic function defined on a domain $(-1, 1) \subset U \subset \mathbb{C}$.

   (iii) Prove that the integral $\int_0^\infty e^{-zt} \sin(t)\, dt$ converges for $\mathrm{Re}(z) > 0$ and defines a holomorphic function in that half-plane. Prove furthermore that the resulting holomorphic function admits an analytic continuation to $\mathbb{C}\backslash\{\pm i\}$.

   (iv) Prove that the series $\sum_{n=0}^\infty z^{(2^n)}$ defines a holomorphic function on the disc $B(0; 1)$ which admits no analytic continuation to any larger domain $B(0; 1) \subsetneq U \subset \mathbb{C}$.

10. Find the Laurent expansion (in powers of z) of $1/(z^2 - 3z + 2)$ in each of the regions:

$$\{z \mid |z| < 1\}; \quad \{z \mid 1 < |z| < 2\}; \quad \{z \bigm| |z| > 2\}.$$

11. Classify the singularities of each of the following functions:

$$\frac{1}{z^2} + \frac{1}{z^2 + 1}, \qquad \frac{z}{\sin z}, \qquad \sin\frac{\pi}{z^2}, \qquad \frac{1}{z^2}\cos\left(\frac{\pi z}{z + 1}\right).$$

12. (Casorati-Weierstrass theorem) Let f be holomorphic on $B(a; r) \setminus \{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in B(a; r)$ with $z_n \neq a$ such that $z_n \to a$ and $f(z_n) \to b$ as $n \to \infty$.

    Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

    [A much harder theorem of Picard says that in any neighbourhood of an essential singularity, a holomorphic function takes *every* complex value except possibly one.]

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Complex Analysis IB: 2015-16 – Sheet 3

1. Let f be a meromorphic function on $\mathbb{C}$ for which $|f(z)| \to \infty$ as $|z| \to \infty$. Show that f cannot have poles at all integer points.

2. Let $g(z) = p(z)/q(z)$ be a rational function with $\deg(q) \geq \deg(p) + 2$. Show that the sum of the residues of g over all its singularities is zero.

3. Prove that the group of conformal automorphisms of the Riemann sphere $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$ is the Möbius group. [*Hint: take an automorphism* g *fixing* 0 *and* ∞ *and consider* $z \mapsto g(z)/z$.]

4. Evaluate the following:

$$(a) \quad \int_0^\pi \frac{d\theta}{4 + \sin^2\theta}; \qquad (b) \quad \int_0^\infty \frac{x^2\,dx}{(x^2+4)^2(x^2+9)};$$
$$(c) \quad \int_0^\infty \sin x^2\,dx; \qquad (d) \quad \int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx.$$

5. For $-1 < \alpha < 1$ and $\alpha \neq 0$, compute

$$\int_0^\infty \frac{x^\alpha}{1 + x + x^2}\,dx.$$

6. Establish the following refinement of the Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree n, and let $A = \max\{|a_i|,\ 0 \leq i \leq n-1\}$. Then $p(z)$ has n roots (counted with multiplicity) in the disk $\{|z| < A + 1\}$.

7. Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\operatorname{Im} p(z) = 0$. Calculate $\operatorname{Re} p(z)$ for such z. Sketch the curve $p \circ \gamma$, where $\gamma(t) = e^{2\pi i t}$, and hence determine the number of z (counted with multiplicity) such that $|z| < 1$ and $p(z) = x$ for each $x \in \mathbb{R}$.

8. (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. Show that there exists $C > 0$ such that for every N, $|\cot \pi z| < C$ on γ_N.

   (ii) By integrating $\dfrac{\pi \cot \pi z}{z^2 + 1}$ around γ_N, show that

$$\sum_{n=0}^\infty \frac{1}{n^2+1} = \frac{1 + \pi \coth\pi}{2}.$$

   (iii) Evaluate $\sum_{n=0}^\infty (-1)^n/(n^2+1)$.

9. Show that the Taylor expansion of $z/(e^z - 1)$ near the origin has the form

$$1 - \frac{z}{2} + \sum_{k=1}^\infty \frac{(-1)^{k-1}B_k}{(2k)!} z^{2k}$$

   where the numbers B_k (the *Bernoulli numbers*) are rational.

10. Let $w \in \mathbb{C}$, and let $\gamma,\ \delta\colon [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number $I(\sigma; 0)$ about the origin for the closed curve $\sigma(t) = (\delta(t) - w)/(\gamma(t) - w)$, show that $I(\gamma; w) = I(\delta; w)$.

    (ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $B(w; r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in B(w; r)$. Deduce that if γ is a closed curve in $\mathbb{C}$ and U is the complement of γ, then the function $w \mapsto I(\gamma; w)$ is a locally constant function on U.

11. (i) Show that $z^4 + 12z + 1$ has exactly three zeros in the annulus $\{1 < |z| < 4\}$.

    (ii) Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{\, z \mid \mathrm{Re}(z) < 0 \,\}$.

    (iii) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{\, z \mid |z| = 3/2 \,\}$.

12. Show that the equation $z \sin z = 1$ has only real solutions.

    [*Hint: Find the number of real roots in the interval* $[-(n + 1/2)\pi, (n + 1/2)\pi]$ *and compare with the number of zeroes of* $z \sin z - 1$ *in a square box* $\{|\mathrm{Re}(z)|, |\mathrm{Im}(z)| < (n + 1/2)\pi\}$.]

13* (Additional) Let U be a domain, let $f : U \to \mathbb{C}$ be holomorphic and suppose $a \in U$ with $f'(a) \neq 0$. Show that for $r > 0$ sufficiently small,

$$g(w) \;=\; \frac{1}{2\pi i} \int_{|z-a|=r} \frac{z f'(z)}{f(z) - w}\, dz$$

defines a holomorphic function g in a neighbourhood of $f(a)$ which is inverse to f.

The following integrals are *not* part of the question sheet, but may provide a good start for revision or a first port of call for the addicted.

(*i*) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin mx}{x(a^2 + x^2)}\, dx$ where $a,\ m \in \mathbb{R}^+$;

(*ii*) $\displaystyle\int_0^{2\pi} \frac{\cos^3 3t}{1 - 2a\cos t + a^2}\, dt$ where $a \in (0,1)$;

(*iii*) $\displaystyle\int_{-1}^{1} \frac{\sqrt{1 - x^2}}{1 + x^2}\, dx$ ("dog-bone" contour);

(*iv*) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-itx}\, dx,$ where $t \in \mathbb{R}$.

(*v*) By integrating $z/(a - e^{-iz})$ round the rectangle with vertices $\pm\pi$, $\pm\pi + iR$, prove that

$$\int_0^{\pi} \frac{x \sin x}{1 - 2a\cos x + a^2}\, dx = \frac{\pi}{a} \log(1 + a) \quad \text{for } a \in (0,1).$$

(*vi*) Assuming $\alpha \geq 0$ and $\beta \geq 0$ prove that

$$\int_0^{\infty} \frac{\cos \alpha x - \cos \beta x}{x^2}\, dx = \frac{\pi}{2}(\beta - \alpha),$$

and deduce the value of

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx.$$

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