Part IB — Complex Analysis
Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Analytic functions
Complex differentiation and the Cauchy–Riemann equations. Examples. Conformal mappings. Informal discussion of branch points, examples of $\log z$ and z^c. [3]

Contour integration and Cauchy's theorem
Contour integration (for piecewise continuously differentiable curves). Statement and proof of Cauchy's theorem for star domains. Cauchy's integral formula, maximum modulus theorem, Liouville's theorem, fundamental theorem of algebra. Morera's theorem. [5]

Expansions and singularities
Uniform convergence of analytic functions; local uniform convergence. Differentiability of a power series. Taylor and Laurent expansions. Principle of isolated zeros. Residue at an isolated singularity. Classification of isolated singularities. [4]

The residue theorem
Winding numbers. Residue theorem. Jordan's lemma. Evaluation of definite integrals by contour integration. Rouché's theorem, principle of the argument. Open mapping theorem. [4]

Contents

0 Introduction

1 Complex differentiation

1.1 Differentiation

Definition (Open subset). A subset $U \subseteq \mathbb{C}$ is *open* if for any $x \in U$, there is some $\varepsilon > 0$ such that the open ball $B_\varepsilon(x) = B(x; \varepsilon) \subseteq U$.

Definition (Path-connected subset). A subset $U \subseteq \mathbb{C}$ is path-connected if for any $x, y \in U$, there is some $\gamma : [0, 1] \to U$ continuous such that $\gamma(0) = x$ and $\gamma(1) = y$.

Definition (Domain). A *domain* is a non-empty open path-connected subset of $\mathbb{C}$.

Definition (Differentiable function). Let $U \subseteq \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be a function. We say f is *differentiable* at $w \in U$ if

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$$

exists.

Definition (Analytic/holomorphic function). A function f is *analytic* or *holomorphic* at $w \in U$ if f is differentiable on an open neighbourhood $B(w, \varepsilon)$ of w (for some ε).

Definition (Entire function). If $f : \mathbb{C} \to \mathbb{C}$ is defined on all of $\mathbb{C}$ and is holomorphic on $\mathbb{C}$, then f is said to be *entire*.

1.2 Conformal mappings

Definition (Conformal function). Let $f : U \to \mathbb{C}$ be a function holomorphic at $w \in U$. If $f'(w) \neq 0$, we say f is *conformal* at w.

Definition (Conformal equivalence). If U and V are open subsets of $\mathbb{C}$ and $f : U \to V$ is a conformal bijection, then it is a *conformal equivalence*.

Notation. We write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$$

is the upper half plane.

Definition (Simple closed curve). A *simple closed curve* is the image of an injective map $S^1 \to \mathbb{C}$.

Definition (Simply connected). A domain $\mathcal{U} \subseteq \mathbb{C}$ is *simply connected* if every continuous map from the circle $f : S^1 \to \mathcal{U}$ can be extended to a continuous map from the disk $F : \overline{D^2} \to \mathcal{U}$ such that $F|_{\partial \overline{D^2}} = f$. Alternatively, any loop can be continuously shrunk to a point.

1.3 Power series

Definition (Uniform convergence). A sequence (f_n) of functions *converge uniformly* to f if for all $\varepsilon > 0$, there is some N such that $n > N$ implies $|f_n(z) - f(z)| < \varepsilon$ for all z.

1.4 Logarithm and branch cuts

Definition (Branch of logarithm)**.** Let $U \subseteq \mathbb{C}^*$ be an open subset. A *branch of the logarithm* on U is a continuous function $\lambda : U \to \mathbb{C}$ for which $e^{\lambda(z)} = z$ for all $z \in U$.

2 Contour integration

2.1 Basic properties of complex integration

Definition (Path)**.** A *path* in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$.

Definition (Simple path)**.** A path $\gamma : [a, b] \to \mathbb{C}$ is *simple* if $\gamma(t_1) = \gamma(t_2)$ only if $t_1 = t_2$ or $\{t_1, t_2\} = \{a, b\}$.

Definition (Closed path)**.** A path $\gamma : [a, b] \to \mathbb{C}$ is *closed* if $\gamma(a) = \gamma(b)$.

Definition (Contour)**.** A *contour* is a simple closed path which is piecewise C^1, i.e. piecewise continuously differentiable.

Definition (Complex integration)**.** If $\gamma : [a, b] \to U \subseteq \mathbb{C}$ is C^1-smooth and $f : U \to \mathbb{C}$ is continuous, then we define the *integral* of f along γ as

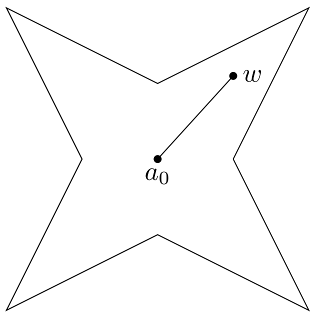
$$\int_\gamma f(z)\ \mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t.$$

By summing over subdomains, the definition extends to piecewise C^1-smooth paths, and in particular contours.

Definition (Antiderivative)**.** Let $U \subseteq \mathbb{C}$ and $f : U \to \mathbb{C}$ be continuous. An *antiderivative* of f is a holomorphic function $F : U \to \mathbb{C}$ such that $F'(z) = f(z)$.

2.2 Cauchy's theorem

Definition (Star-shaped domain)**.** A *star-shaped domain* or *star domain* is a domain U such that there is some $a_0 \in U$ such that the line segment $[a_0, w] \subseteq U$ for all $w \in U$.



Definition (Triangle)**.** A *triangle* in a domain U is what it ought to be — the Euclidean convex hull of 3 points in U, lying wholly in U. We write its boundary as ∂T, which we view as an oriented piecewise C^1 path, i.e. a contour.

2.3 The Cauchy integral formula

Definition (Elementary deformation)**.** Given a pair of C^1-smooth (or piecewise smooth) closed paths $\phi, \psi : [0, 1] \to U$, we say ψ is an elementary deformation of ϕ if there exists convex open sets $C_1, \cdots, C_n \subseteq U$ and a division of the interval $0 = x_0 < x_1 < \cdots < x_n = 1$ such that on $[x_{i-1}, x_i]$, both $\phi(t)$ and $\psi(t)$ belong to C_i.

2.4 Taylor's theorem

2.5 Zeroes

Definition (Order of zero)**.** Let $f : B(a, r) \to \mathbb{C}$ be holomorphic. Then we know we can write

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

as a convergent power series. Then either all $c_n = 0$, in which case $f = 0$ on $B(a, r)$, or there is a least N such that $c_N \neq 0$ (N is just the smallest n such that $f^{(n)}(a) \neq 0$).

If $N > 0$, then we say f has a *zero of order* N.

Definition (Analytic continuiation)**.** Let $U_0 \subseteq U \subseteq \mathbb{C}$ be domains, and $f : U_0 \to \mathbb{C}$ be holomorphic. An *analytic continuation* of f is a holomorphic function $h : U \to \mathbb{C}$ such that $h|_{U_0} = f$, i.e. $h(z) = f(z)$ for all $z \in U_0$.

2.6 Singularities

Definition (Isolated singularity)**.** Given a domain U and $z_0 \in U$, and $f : U \setminus \{z_0\} \to \mathbb{C}$ holomorphic, we say z_0 is an *isolated singularity* of f.

Definition (Removable singularity)**.** A singularity z_0 of f is a *removable singularity* if f is bounded near z_0.

Definition (Pole)**.** A singularity z_0 is a *pole of order* k of f if $|f(z)| \to \infty$ as $z \to z_0$ and one can write

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

with $g : U \to \mathbb{C}$, $g(z_0) \neq 0$.

Definition (Isolated essential singularity)**.** An isolated singularity is an *isolated essential singularity* if it is neither removable nor a pole.

Definition (Meromorphic function)**.** If U is a domain and $S \subseteq U$ is a finite or discrete set, a function $f : U \setminus S \to \mathbb{C}$ which is holomorphic and has (at worst) poles on S is said to be *meromorphic* on U.

2.7 Laurent series

Definition (Principal part)**.** If $f : B(a, r) \setminus \{a\} \to \mathbb{C}$ is holomorphic and if f has Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n,$$

then the *principal part* of f at a is

$$f_{\text{principal}} = \sum_{n=-\infty}^{-1} c_n (z - a)^n.$$

3 Residue calculus

3.1 Winding numbers

Definition (Residue)**.** Let $f : B(a, r) \setminus \{a\} \to \mathbb{C}$ be holomorphic, with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n.$$

Then the *residue* of f at a is

$$\operatorname{Res}(f, a) = \operatorname{Res}_f(a) = c_{-1}.$$

Definition (Winding number)**.** Given a continuous path $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(a) = \gamma(b)$ and $w \not\in \operatorname{image}(\gamma)$, the *winding number* of γ about w is

$$\frac{\theta(b) - \theta(a)}{2\pi},$$

where $\theta : [a, b] \to \mathbb{R}$ is a continuous function as above. This is denoted by $I(\gamma, w)$ or $n_\gamma(W)$.

3.2 Homotopy of closed curves

Definition (Homotopy of closed curves)**.** Let $U \subseteq \mathbb{C}$ be a domain, and let $\phi : [a, b] \to U$ and $\psi : [a, b] \to U$ be piecewise C^1-smooth closed paths. A *homotopy* from $\phi : \psi$ is a continuous map $F : [0, 1] \times [a, b] \to U$ such that

$$F(0, t) = \phi(t), \quad F(1, t) = \psi(t),$$

and moreover, for all $s \in [0, t]$, the map $t \mapsto F(s, t)$ viewed as a map $[a, b] \to U$ is closed and piecewise C^1-smooth.

Definition (Simply connected domain)**.** A domain U is *simply connected* if every C^1 smooth closed path is homotopic to a constant path.

3.3 Cauchy's residue theorem

3.4 Overview

3.5 Applications of the residue theorem

3.6 Rouchés theorem

Definition (Local degree)**.** Let $f : B(a, r) \to \mathbb{C}$ be holomorphic and non-constant. Then the *local degree* of f at a, written $\deg(f, a)$ is the order of the zero of $f(z) - f(a)$ at a.