# Part IB — Variational Principles

## Theorems with proof

Based on lectures by P. K. Townsend

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Easter 2015

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Stationary points for functions on $\mathbb{R}^n$. Necessary and sufficient conditions for minima and maxima. Importance of convexity. Variational problems with constraints; method of Lagrange multipliers. The Legendre Transform; need for convexity to ensure invertibility; illustrations from thermodynamics. [4]

The idea of a functional and a functional derivative. First variation for functionals, Euler-Lagrange equations, for both ordinary and partial differential equations. Use of Lagrange multipliers and multiplier functions. [3]

Fermat's principle; geodesics; least action principles, Lagrange's and Hamilton's equations for particles and fields. Noether theorems and first integrals, including two forms of Noether's theorem for ordinary differential equations (energy and momentum, for example). Interpretation in terms of conservation laws. [3]

Second variation for functionals; associated eigenvalue problem. [2]

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# 0 Introduction

# 1 Multivariate calculus

## 1.1 Stationary points

## 1.2 Convex functions

### 1.2.1 Convexity

### 1.2.2 First-order convexity condition

**Corollary.** A stationary point of a convex function is a global minimum. There can be more than one global minimum (e.g. a constant function), but there is at most one if the function is strictly convex.

*Proof.* Given $\mathbf{x}_0$ such that $\nabla f(\mathbf{x}_0) = \mathbf{0}$, (†) implies that for any $\mathbf{y}$,

$$f(\mathbf{y}) \geq f(\mathbf{x}_0) + (\mathbf{y} - \mathbf{x}_0) \cdot \nabla f(\mathbf{x}_0) = f(\mathbf{x}_0). \qquad \square$$

### 1.2.3 Second-order convexity condition

## 1.3 Legendre transform

**Lemma.** $f^*$ is always convex.

*Proof.*

$$\begin{aligned} f^*((1-t)\mathbf{p} + t\mathbf{q}) &= \sup_{\mathbf{x}} \big[((1-t)\mathbf{p}\cdot\mathbf{x} + t\mathbf{q}\cdot\mathbf{x} - f(\mathbf{x})\big]. \\ &= \sup_{\mathbf{x}} \big[(1-t)(\mathbf{p}\cdot\mathbf{x} - f(\mathbf{x})) + t(\mathbf{q}\cdot\mathbf{x} - f(\mathbf{x}))\big] \\ &\leq (1-t)\sup_{\mathbf{x}}[\mathbf{p}\cdot\mathbf{x} - f(\mathbf{x})] + t\sup_{\mathbf{x}}[\mathbf{q}\cdot\mathbf{x} - f(\mathbf{x})] \\ &= (1-t)f^*(\mathbf{p}) + tf^*(\mathbf{q}) \end{aligned}$$

Note that we cannot immediately say that $f^*$ is convex, since we have to show that the domain is convex. But by the above bounds, $f^*((1-t)\mathbf{p}+t\mathbf{q})$ is bounded by the sum of two finite terms, which is finite. So $(1-t)\mathbf{p} + t\mathbf{q}$ is also in the domain of $f^*$. $\square$

**Theorem.** If $f$ is convex, differentiable with Legendre transform $f^*$, then $f^{**} = f$.

*Proof.* We have $f^*(\mathbf{p}) = (\mathbf{p}\cdot\mathbf{x}(\mathbf{p}) - f(\mathbf{x}(\mathbf{p}))$ where $\mathbf{x}(\mathbf{p})$ satisfies $\mathbf{p} = \nabla f(\mathbf{x}(\mathbf{p}))$.

Differentiating with respect to $\mathbf{p}$, we have

$$\begin{aligned} \nabla_i f^*(\mathbf{p}) &= x_i + p_j \nabla_i x_j(\mathbf{p}) - \nabla_i x_j(\mathbf{p}) \nabla_j f(\mathbf{x}) \\ &= x_i + p_j \nabla_i x_j(\mathbf{p}) - \nabla_i x_j(\mathbf{p}) p_j \\ &= x_i. \end{aligned}$$

So

$$\nabla f^*(\mathbf{p}) = \mathbf{x}.$$

This means that the conjugate variable of $\mathbf{p}$ is our original $\mathbf{x}$. So

$$\begin{aligned} f^{**}(\mathbf{x}) &= (\mathbf{x}\cdot\mathbf{p} - f^*(\mathbf{p}))|_{\mathbf{p}=\mathbf{p}(\mathbf{x})} \\ &= \mathbf{x}\cdot\mathbf{p} - (\mathbf{p}\cdot\mathbf{x} - f(\mathbf{x})) \\ &= f(\mathbf{x}). \end{aligned} \qquad \square$$

## 1.4 Lagrange multipliers

# 2 Euler-Lagrange equation

## 2.1 Functional derivatives

## 2.2 First integrals

## 2.3 Constrained variation of functionals

# 3 Hamilton's principle

## 3.1 The Lagrangian

**Law** (Hamilton's principle)**.** The actual path $\xi(t)$ taken by a particle is the path that makes the action $S$ stationary.

## 3.2 The Hamiltonian

## 3.3 Symmetries and Noether's theorem

**Theorem** (Noether's theorem)**.** For every continuous symmetry of $F[x]$, the solutions (i.e. the stationary points of $F[x]$) will have a corresponding conserved quantity.

# 4 Multivariate calculus of variations

# 5 The second variation

## 5.1 The second variation

## 5.2 Jacobi condition for local minima of $F[x]$