A1a **Variational Principles: Example Sheet 1** Easter term 2015

Corrections and suggestions should be emailed to P.K.Townsend@damtp.cam.ac.uk.

1. At how many points does the function

$$\phi(x_1, x_2, x_3) = \tfrac{1}{4}(x_1^4 + x_2^4 + x_3^4) - x_2x_3 - x_3x_1 - x_1x_2$$

take its minimum value? Show that this least value is -3. By considering the eigenvalues of the Hessian, show also that ϕ has one saddle point, at which the surface of vanishing ϕ is tangent to a double cone of semi-angle $\arctan\sqrt{2}$.

2. Show that

(i) x^2/y is convex on the upper half plane $(x, y) : y > 0$.

(ii) Show that if $f(x)$ is convex then the function $yf(y^{-1}x)$ is convex on $(x, y) : y > 0$.

3. Find the Legendre transform of $f(x) = e^x$, (giving its domain also). Find the Legendre transform of $f(x) = a^{-1}x^a, a > 1$ defined on $x > 0$, and hence deduce Young's inequality

$$xy \;\le\; \frac{x^a}{a} + \frac{y^b}{b}\,, \qquad \frac{1}{a} + \frac{1}{b} = 1\,.$$

4. For an ideal gas, the internal energy $U = U(S, V)$ as a function of entropy and volume is

$$U = U_0 + \alpha nRT_0\Big[\big(\frac{V_0}{V}\big)^{\frac{1}{\alpha}} e^{\frac{S-S_0}{\alpha nR}} - 1\Big]$$

for some constants $U_0, T_0, V_0, S_0, \alpha, n, R$. Calculate the Helmholtz free energy $F = F(T, V)$ defined by $F(T, V) = \min_S(U(S, V) - TS)$.

5. The area A of a triangle with sides a, b, c is given by

$$A = \sqrt{[s(s-a)(s-b)(s-c)]}, \quad \text{where } s = \tfrac{1}{2}(a + b + c).$$

(i) Show that of all triangles of given perimeter $2s$, the triangle of largest area is equilateral.

(ii) Find (in terms of the perimeter) the largest possible area of a right-angled triangle of given perimeter.

6. Find the maximum volume of a rectangular parallelepiped inscribed inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,.$$

7. Let (θ, ϕ) be the standard angular coordinates on the unit sphere. A function $\phi(\theta)$ defines a path on this sphere. Given that θ increases by $\delta\theta$ over a short segment of this path, show that the length of the path segment is $\sqrt{1 + (\phi' \sin\theta)^2}\ \delta\theta$ to first order in $\delta\theta$. Hence find a functional $L[\phi]$ for the total length of a path between any two points on the unit sphere. Use your result to show that the paths of mimimal length are segments of great circles.

8. A soap film is bounded by two circular wires at $r = a,\ z = \pm b$ in cylindrical polar coordinates (r, θ, z). Given that the soap surface is cylindrically symmetric, show that the equation of the surface of minimal area is

$$r = c\cosh\left(z/c\right)$$

where c satisfies the condition $a/c = \cosh\left(b/c\right)$. Show graphically that this condition has no solution for c if b/a is larger that a certain critical ratio. What happens to the soap surface as b/a is increased from below this ratio to above it?

9. It has been suggested that Crossrail, the new rail connection under London, should include a frictionless tunnel in which fuel-less trains can run under gravity. The trains are released from rest at the point of departure (Stratford) and are then allowed to run freely until arriving at their destination (Acton) at the same level. Assuming that the acceleration g due to gravity is uniform, show that the minimum travel time is $\sqrt{2\pi\ell/g}$, where ℓ is the horizontal distance between the departure and arrival points. Comment on the quality of the likely travel experience at the departure and arrival points.

10. In an optical medium filling the region $0 < y < h$, the speed of light is

$$c(y) = \frac{c_0}{\sqrt{1-ky}}\,, \qquad 0 < k < 1/h\,.$$

Show that the paths of light rays in the medium are parabolic. Show also that if a ray enters the medium at $(-x_0, 0)$ and leaves it at $(x_0, 0)$ then

$$(kx_0)^2 = 4ky_0(1-ky_0)\,,$$

where y_0 $(< h)$ is the greatest value of y attained on the ray path.

11. Determine all functions $u(x)$ that extremize the functional

$$I[u] = \int_{-\infty}^{\infty} \left\{ \frac{1}{2} u'^2 + (1-\cos u) \right\} dx \qquad (*)$$

subject to the boundary conditions

$$\lim_{x\to-\infty} u(x) = 0\,, \qquad \lim_{x\to\infty} u(x) = 2\pi\,.$$

For $u(x)$ satisfying these boundary conditions, show that

$$I[u] = \frac{1}{2}\int_{-\infty}^{\infty} \left(u' - 2\sin(u/2)\right)^2 dx \,+\, 8\,.$$

Deduce that $I[u] \le 8$ for such functions, and write down a *first-order* differential equation that u must satisfy in order to realise this lower bound. How is this differential equation related to the second-order Euler-Lagrange equation for the function $I[u]$ of $(*)$?

12. *Dido's problem.* An area A of a field is enclosed by a length ℓ of flexible fencing with its ends attached a distance a apart on a straight wall, where $a < \ell < \frac{1}{2}\pi a$. Show that A is maximised, for fixed ℓ, by the arc of a circle, and derive an equation relating the radius of the circle to the location of its centre. Comment on the case of $\ell > \frac{1}{2}\pi a$.

13. A uniform cable of fixed length, suspended between the two points $(-a, b)$ and (a, b) has potential energy

$$V = \int_{-a}^{a} y\sqrt{1+y'^2}\,dx\,.$$

Write down a functional for the total length and then use the Lagrange multiplier method to show that the curve $y(x)$ of minumum energy assumed by the cable is a catenary:

$$y - y_0 = c\cosh\left(\frac{x-x_0}{c}\right)\,,$$

where x_0, y_0 and c are constants. Find an equation for c and show that it has a unique positive solution.

A1a **Variational Principles: Example Sheet 2** Easter term 2014

Corrections and suggestions should be emailed to `p.k.townsend@damtp.cam.ac.uk`.

1. Obtain the Euler-Lagrange equation for the function $x(t)$ that makes stationary the functional

$$F[x] = \int_{t_1}^{t_2} f(t, x(t), \dot{x}(t), \ddot{x}(t))\, dt$$

for fixed values of both $x(t)$ and $\dot{x}(t)$ at both $t = t_1$ and $t = t_2$.

Find the function $x(t)$ that minimises the integral $\int_1^2 t^4[\ddot{x}(t)]^2 dt$ subject to boundary conditions

$$x(1) = 1\,, \qquad \dot{x}(1) = -2\,, \qquad x(2) = \frac{1}{4}\,, \qquad \dot{x}(2) = -\frac{1}{4}\,.$$

Include a demonstration that it is a global minimiser (not just a stationary point) for the integral.

2. A simple closed curve in the x-y plane is given, in terms of an arbitrary angular parameter θ by the functions $x(\theta)$ and $y(\theta)$ for $0 \le \theta < 2\pi$. The area enclosed by the curve is

$$A[x, y] = \frac{1}{2}\int_0^{2\pi} (xy' - yx')\, d\theta\,.$$

Use this expression to find the curve that maximises the enclosed area for fixed length.

3. Using the Lagrange multiplier method, write down the Euler-Lagrange equations associated to the problem of minimising the functional

$$I[\psi] = \int_{-\infty}^{+\infty} \left(\psi'^2 + x^2\psi^2\right) dx$$

subject to the normalization condition $\int \psi^2 dx = 1$. Given that $x\psi(x)^2 \to 0$ as $x \to \pm\infty$, show that

$$I[\psi] = 1 + \int_{-\infty}^{+\infty} (\psi' + x\psi)^2 dx\,,$$

and hence deduce that $I \ge 1$. Show that equality holds for a function ψ that you should give explicitly. Verify that it satisfies the Euler-Lagrange equation for an appropriate value of the Lagrange multiplier.

4. Let $\mathbf{x}(t) \in \mathbb{R}^3$ be a curve which is constrained to lie on the sphere $S^2 = \{\mathbf{x} : |\mathbf{x}| = 1\}$. Use the Lagrange multiplier function formalism to obtain the following Euler-Lagrange equation

$$\ddot{\mathbf{x}} + |\dot{\mathbf{x}}|^2\mathbf{x} = 0$$

for the problem of minimising $I[\mathbf{x}] = \int |\dot{\mathbf{x}}|^2 dt$ amongst curves satisfying the constraint $\mathbf{x}(t) \in S^2$. Show that the solutions of the Euler-Lagrange equation lie on a plane through the origin (they are great circles.)

5. A particle of mass m is constrained to roll on the inside of a smooth upturned hemispherical bowl of radius a. The Lagrangian describing the motion is

$$L = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2(\sin^2\theta)\dot{\phi}^2 + mga\cos\theta\,,$$

where g is the acceleration due to gravity, and θ and ϕ are the usual spherical angles (with θ measured relative to the downward vertical). Find two constants of the motion.

Find the two momenta p_θ and p_ϕ, and write down the Hamiltonian of the system. What do Hamilton's equations become in this case?

6. Hamilton's Principle is applicable to the *relativistic* dynamics of a charged particle in an electromagnetic field. The appropriate choice of Lagrangian $L[\mathbf{x}(t), \dot{\mathbf{x}}(t), t]$ for a particle of rest-mass m and charge q in a given electric potential $\phi(t, \mathbf{x})$ and magnetic vector potential $\mathbf{A}(t, \mathbf{x})$ is

$$L = -mc^2\sqrt{1 - |\mathbf{v}|^2/c^2} - q\phi + q\mathbf{v}\cdot\mathbf{A}\,,$$

where $\mathbf{v} = \dot{\mathbf{x}}(t)$. Verify that the Euler-Lagrange equations yield the equation of motion

$$\frac{d}{dt}(m_0\gamma\mathbf{v}) = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\,, \qquad \gamma = (1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}}\,,$$

where $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ is the electric field and $\mathbf{B} = \nabla\times\mathbf{A}$ the magnetic field.

7. Obtain the Euler-Lagrange equations associated with the functionals

(i) $I[u] = \int(\frac{1}{2}u_t^2 - F(u_x))dxdt$,

(ii) $I[u] = \int(|\nabla u|^2 + e^{2u})dxdy$,

(iii) * $I[u] = \int(\det Du)dxdy$, where $u : \mathbb{R}^2 \to \mathbb{R}^2$, and $\det Du$ means the Jacobian determinant. What is unusual about this example?

8. The mass density $\rho(t, \mathbf{x})$ and velocity field $\mathbf{v}(t, \mathbf{x})$ of a compressible fluid are constrained by conservation of mass to satisfy the continuity equation

$$\dot{\rho} + \nabla\cdot(\rho\mathbf{v}) = 0\,. \qquad (*)$$

Given that the energy density of the fluid is $u(\rho)$, the action (for inviscid irrotational flow) is

$$S[\rho, \mathbf{v}, \phi] = \int d^3x \left\{\frac{1}{2}\rho|\mathbf{v}|^2 - u(\rho) + \phi\left[\dot{\rho} + \nabla\cdot(\rho\mathbf{v})\right]\right\}\,,$$

where $\phi(t, \mathbf{x})$ is a Lagrange multiplier field imposing the continuity condition $(*)$. Find the Euler-Lagrange equations for this action. Show that they imply $\mathbf{v} = \nabla\phi$ (so ϕ is the velocity potential). Given that the fluid pressure $P(t, \mathbf{x})$ satisfies

$$\nabla P = \rho\nabla h(t, \mathbf{x})\,, \qquad h = u'(\rho)\,,$$

deduce Euler's equation for inviscid irrotational flow:

$$\rho\left[\dot{\mathbf{v}} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = -\nabla P\,.$$

9. For the length functional for curves on the sphere $I[\phi] = \int_a^b (1 + \phi'^2\sin^2\theta)^{\frac{1}{2}}d\theta$, compute the second variation of I and show that it is positive.

10. For $I[y] = \int_a^b (y'^2 + y^4)dx$ with $y(a) = \alpha$, $y(b) = \beta$, find the Euler-Lagrange equation and the second variation. For the case $\alpha = 0 = \beta$ write down the solution of the Euler-Lagrange equation and the second variation explicitly, and show that the second variation is strictly positive.

11. Consider $I[y] = \int_0^1 \left(\frac{1}{2}y'^2 + F(y)\right)dx$ with $y(0) = 0 = y(1)$ and suppose $F'(0) = 0$. Write down the associated Euler-Lagrange equation, and show that $y_0(x) = 0$ is a solution. Find the second variation, and deduce the range of values of $F''(0)$ for the second variation to be positive. [NB this includes a range of negative values of $F''(0)$].