Part IB — Variational Principles
Definitions

Based on lectures by P. K. Townsend
Notes taken by Dexter Chua

Easter 2015

Stationary points for functions on $\mathbb{R}^n$. Necessary and sufficient conditions for minima and maxima. Importance of convexity. Variational problems with constraints; method of Lagrange multipliers. The Legendre Transform; need for convexity to ensure invertibility; illustrations from thermodynamics. [4]

The idea of a functional and a functional derivative. First variation for functionals, Euler-Lagrange equations, for both ordinary and partial differential equations. Use of Lagrange multipliers and multiplier functions. [3]

Fermat's principle; geodesics; least action principles, Lagrange's and Hamilton's equations for particles and fields. Noether theorems and first integrals, including two forms of Noether's theorem for ordinary differential equations (energy and momentum, for example). Interpretation in terms of conservation laws. [3]

Second variation for functionals; associated eigenvalue problem. [2]

Contents

0 Introduction

1 Multivariate calculus

1.1 Stationary points

Definition (Stationary points). *Stationary points* are points in $\mathbb{R}^n$ for which $\nabla f = \mathbf{0}$, i.e.

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

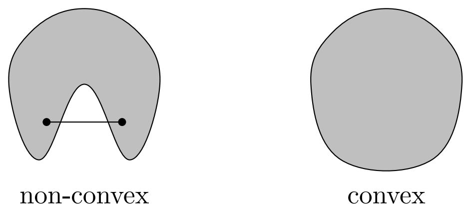
Definition (Hessian matrix). The *Hessian matrix* is

$$H_{ij}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

1.2 Convex functions

1.2.1 Convexity

Definition (Convex set). A set $S \subseteq \mathbb{R}^n$ is *convex* if for any distinct $\mathbf{x}, \mathbf{y} \in S, t \in (0, 1)$, we have $(1-t)\mathbf{x} + t\mathbf{y} \in S$. Alternatively, any line joining two points in S lies completely within S.



Definition (Convex function). A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if

(i) The domain $D(f)$ is convex

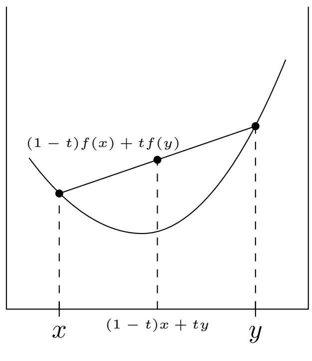
(ii) The function f lies below (or on) all its chords, i.e.

$$f((1-t)\mathbf{x} + t\mathbf{y}) \leq (1-t)f(\mathbf{x}) + tf(\mathbf{y}) \qquad (*)$$

for all $\mathbf{x}, \mathbf{y} \in D(f), t \in (0, 1)$.

A function is *strictly convex* if the inequality is strict, i.e.

$$f((1-t)\mathbf{x} + t\mathbf{y}) < (1-t)f(\mathbf{x}) + tf(\mathbf{y}).$$



A function f is (strictly) concave iff $-f$ is (strictly) convex.

1.2.2 First-order convexity condition

1.2.3 Second-order convexity condition

1.3 Legendre transform

Definition (Legendre transform)**.** Given a function $f : \mathbb{R}^n \to \mathbb{R}$, its *Legendre transform* f^* (the "conjugate" function) is defined by

$$f^*(\mathbf{p}) = \sup_{\mathbf{x}}(\mathbf{p} \cdot \mathbf{x} - f(\mathbf{x})),$$

The domain of f^* is the set of $\mathbf{p} \in \mathbb{R}^n$ such that the supremum is finite. $\mathbf{p}$ is known as the conjugate variable.

1.4 Lagrange multipliers

2 Euler-Lagrange equation

2.1 Functional derivatives

Definition (Functional)**.** A *functional* is a function that takes in another real-valued function as an argument. We usually write them as $F[x]$ (square brackets), where $x = x(t) : \mathbb{R} \to \mathbb{R}$ is a real function. We say that $F[x]$ is a functional of the function $x(t)$.

Definition (Functional derivative)**.**

$$\frac{\delta F[x]}{\delta x} = \frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right)$$

is the *functional derivative* of $F[x]$.

Definition (Euler-Lagrange equation)**.** The *Euler-Lagrange* equation is

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0$$

for $\alpha \leq t \leq \beta$.

2.2 First integrals

2.3 Constrained variation of functionals

3 Hamilton's principle

3.1 The Lagrangian

3.2 The Hamiltonian

Definition (Hamiltonian)**.** The *Hamiltonian* of a system is the Legendre transform of the Lagrangian:

$$H(\mathbf{x}, \mathbf{p}) = \mathbf{p} \cdot \dot{\mathbf{x}} - L(\mathbf{x}, \dot{\mathbf{x}}),$$

where $\dot{\mathbf{x}}$ is a function of $\mathbf{p}$ that is the solution to $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}}$.

$\mathbf{p}$ is the *conjugate momentum* of $\mathbf{x}$. The space containing the variables $\mathbf{x}, \mathbf{p}$ is known as the *phase space*.

3.3 Symmetries and Noether's theorem

Definition (Symmetry)**.** If $F^*[x^*] = F[x]$ for all x, α and β, then the transformation $*$ is a *symmetry*.

4 Multivariate calculus of variations

5 The second variation

5.1 The second variation

5.2 Jacobi condition for local minima of $F[x]$