

Part IB — Optimisation

Definitions

Based on lectures by F. A. Fischer

Notes taken by Dexter Chua

Easter 2015

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Lagrangian methods

General formulation of constrained problems; the Lagrangian sufficiency theorem. Interpretation of Lagrange multipliers as shadow prices. Examples. [2]

Linear programming in the nondegenerate case

Convexity of feasible region; sufficiency of extreme points. Standardization of problems, slack variables, equivalence of extreme points and basic solutions. The primal simplex algorithm, artificial variables, the two-phase method. Practical use of the algorithm; the tableau. Examples. The dual linear problem, duality theorem in a standardized case, complementary slackness, dual variables and their interpretation as shadow prices. Relationship of the primal simplex algorithm to dual problem. Two person zero-sum games. [6]

Network problems

The Ford-Fulkerson algorithm and the max-flow min-cut theorems in the rational case. Network flows with costs, the transportation algorithm, relationship of dual variables with nodes. Examples. Conditions for optimality in more general networks; *the simplex-on-a-graph algorithm*. [3]

Practice and applications

Efficiency of algorithms. The formulation of simple practical and combinatorial problems as linear programming or network problems. [1]

Contents

1	Introduction and preliminaries	3
1.1	Constrained optimization	3
1.2	Review of unconstrained optimization	3
2	The method of Lagrange multipliers	4
2.1	Complementary Slackness	4
2.2	Shadow prices	4
2.3	Lagrange duality	4
2.4	Supporting hyperplanes and convexity	4
3	Solutions of linear programs	5
3.1	Linear programs	5
3.2	Basic solutions	5
3.3	Extreme points and optimal solutions	5
3.4	Linear programming duality	5
3.5	Simplex method	5
3.5.1	The simplex tableau	5
3.5.2	Using the Tableau	5
3.6	The two-phase simplex method	5
4	Non-cooperative games	6
4.1	Games and Solutions	6
4.2	The minimax theorem	6
5	Network problems	7
5.1	Definitions	7
5.2	Minimum-cost flow problem	7
5.3	The transportation problem	7
5.4	The maximum flow problem	7

1 Introduction and preliminaries

1.1 Constrained optimization

Definition (Constrained optimization). The general problem is of *constrained optimization* is

$$\text{minimize } f(x) \text{ subject to } h(x) = b, x \in X$$

where $x \in \mathbb{R}^n$ is the *vector of decision variables*, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the *objective function*, $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $b \in \mathbb{R}^m$ are the *functional constraints*, and $X \subseteq \mathbb{R}^n$ is the *regional constraint*.

Definition (General and standard form). The *general form* of a linear program is

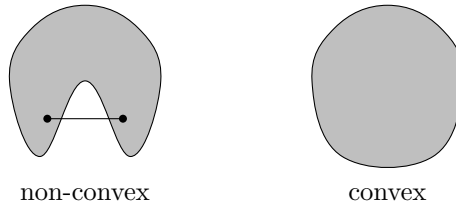
$$\text{minimize } c^T x \text{ subject to } Ax \geq b, x \geq 0$$

The *standard form* is

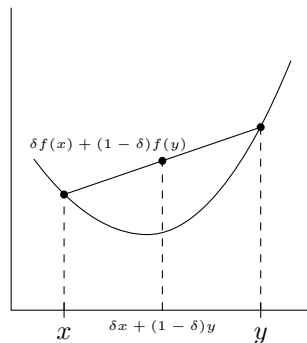
$$\text{minimize } c^T x \text{ subject to } Ax = b, x \geq 0.$$

1.2 Review of unconstrained optimization

Definition (Convex region). A region $S \subseteq \mathbb{R}^n$ is *convex* iff for all $\delta \in [0, 1]$, $x, y \in S$, we have $\delta x + (1 - \delta)y \in S$. Alternatively, If you take two points, the line joining them lies completely within the region.



Definition (Convex function). A function $f : S \rightarrow \mathbb{R}$ is *convex* if S is convex, and for all $x, y \in S$, $\delta \in [0, 1]$, we have $\delta f(x) + (1 - \delta)f(y) \geq f(\delta x + (1 - \delta)y)$.



A function is *concave* if $-f$ is convex. Note that a function can be neither concave nor convex.

Definition (Positive-semidefinite). A matrix H is *positive semi-definite* if $v^T H v \geq 0$ for all $v \in \mathbb{R}^n$.

2 The method of Lagrange multipliers

Definition (Lagrangian). The *Lagrangian* of a constraint (P) is defined as

$$L(x, \lambda) = f(x) - \lambda^T (h(x) - b).$$

for $\lambda \in \mathbb{R}^m$. λ is known as the *Lagrange multiplier*.

2.1 Complementary Slackness

2.2 Shadow prices

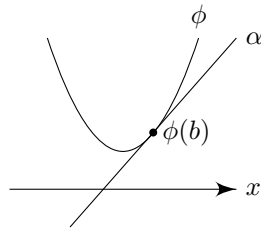
2.3 Lagrange duality

Definition (Strong duality). (P) and (D) are said to satisfy *strong duality* if

$$\sup_{\lambda \in Y} g(\lambda) = \inf_{x \in X(b)} f(x).$$

2.4 Supporting hyperplanes and convexity

Definition (Supporting hyperplane). A hyperplane $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ is *supporting* to ϕ at b if α intersects ϕ at b and $\phi(c) \geq \alpha(c)$ for all c .



3 Solutions of linear programs

3.1 Linear programs

3.2 Basic solutions

Definition (Extreme point). An *extreme point* $x \in S$ of a convex set S is a point that cannot be written as a convex combination of two distinct points in S , i.e. if $y, z \in S$ and $\delta \in (0, 1)$ satisfy

$$x = \delta y + (1 - \delta)z,$$

then $x = y = z$.

Definition (Basic solution and basis). A solution $x \in \mathbb{R}^n$ is *basic* if it has at most m non-zero entries (out of n), i.e. if there exists a set $B \subseteq \{1, \dots, n\}$ with $|B| = m$ such that $x_i = 0$ if $i \notin B$. In this case, B is called the *basis*, and x_i are the *basic variables* if $i \in B$.

Definition (Non-degenerate solutions). A basic solution is *non-degenerate* if it has exactly m non-zero entries.

Definition (Basic feasible solution). A basic solution x is *feasible* if it satisfies $x \geq 0$.

3.3 Extreme points and optimal solutions

3.4 Linear programming duality

3.5 Simplex method

3.5.1 The simplex tableau

3.5.2 Using the Tableau

3.6 The two-phase simplex method

4 Non-cooperative games

4.1 Games and Solutions

Definition (Bimatrix game). A two-player game, or *bimatrix game*, is given by two matrices $P, Q \in \mathbb{R}^{m \times n}$. Player 1, or the *row player*, chooses a row $i \in \{1, \dots, m\}$, while player 2, the *column player*, chooses a column $j \in \{1, \dots, n\}$. These are selected without knowledge of the other player's decisions. The two players then get payoffs P_{ij} and Q_{ij} respectively.

Definition (Strategy). Players are allowed to play randomly. The set of *strategies* the row player can have is

$$X = \{x \in \mathbb{R}^m : x \geq 0, \sum x_i = 1\}$$

and the column player has strategies

$$Y = \{y \in \mathbb{R}^n : y \geq 0, \sum y_i = 1\}$$

Each vector corresponds to the probabilities of selecting each row or column.

A strategy profile $(x, y) \in X \times Y$ induces a lottery, and we write $p(x, y) = x^T P y$ for the expected payoff of the row player.

If $x_i = 1$ for some i , i.e. we always pick i , we call x a *pure strategy*.

Definition (Best response and equilibrium). A strategy $x \in X$ is a *best response* to $y \in Y$ if for all $x' \in X$

$$p(x, y) \geq p(x', y)$$

A pair (x, y) is an *equilibrium* if x is the best response against y and y is a best response against x .

4.2 The minimax theorem

Definition (Zero-sum game). A bimatrix game is a *zero-sum game*, or matrix game, if $q_{ij} = -p_{ij}$ for all i, j , i.e. the total payoff is always 0.

Definition (Value). The *value* of the matrix game with payoff matrix P is

$$v = \max_{x \in X} \min_{y \in Y} p(x, y) = \min_{y \in Y} \max_{x \in X} p(x, y).$$

5 Network problems

5.1 Definitions

Definition (Directed graph/network). A *directed graph* or *network* is a pair $G = (V, E)$, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. If $(u, v) \in E$, we say there is an edge from u to v .

Definition (Degree). The degree of a vertex $u \in V$ is the number of $v \in V$ such that $(u, v) \in E$ or $(v, u) \in E$.

Definition (Walk). An *walk* from $u \in V$ to $v \in V$ is a sequence of vertices $u = v_1, \dots, v_k = v$ such that $(v_i, v_{i+1}) \in E$ for all i . An *undirected walk* allows $(v_i, v_{i+1}) \in E$ or $(v_{i+1}, v_i) \in E$, i.e. we are allowed to walk backwards.

Definition (Path). A path is a walk where v_1, \dots, v_k are pairwise distinct.

Definition (Cycle). A cycle is a walk where v_1, \dots, v_{k-1} are pairwise distinct and $v_1 = v_k$.

Definition (Connected graph). A graph is *connected* if for any pair of vertices, there is an undirected path between them.

Definition (Tree). A *tree* is a connected graph without (undirected) cycles.

Definition (Spanning tree). The *spanning tree* of a graph $G = (V, E)$ is a tree (V', E') with $V' = V$ and $E' \subseteq E$.

5.2 Minimum-cost flow problem

5.3 The transportation problem

5.4 The maximum flow problem

Definition (Cut). Suppose $G = (V, E)$ with capacities C_{ij} for $(i, j) \in E$. A *cut* of G is a partition of V into two sets.

For $S \subseteq V$, the *capacity* of the cut $(S, V \setminus S)$ is

$$C(S) = \sum_{(i,j) \in (S \times (V \setminus S)) \cap E} C_{ij},$$

All this clumsy notation says is that we add up the capacities of all edges from S to $V \setminus S$.