

Part IB — Metric and Topological Spaces

Theorems with proof

Based on lectures by J. Rasmussen

Notes taken by Dexter Chua

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Metrics

Definition and examples. Limits and continuity. Open sets and neighbourhoods. Characterizing limits and continuity using neighbourhoods and open sets. [3]

Topology

Definition of a topology. Metric topologies. Further examples. Neighbourhoods, closed sets, convergence and continuity. Hausdorff spaces. Homeomorphisms. Topological and non-topological properties. Completeness. Subspace, quotient and product topologies. [3]

Connectedness

Definition using open sets and integer-valued functions. Examples, including intervals. Components. The continuous image of a connected space is connected. Path-connectedness. Path-connected spaces are connected but not conversely. Connected open sets in Euclidean space are path-connected. [3]

Compactness

Definition using open covers. Examples: finite sets and $[0, 1]$. Closed subsets of compact spaces are compact. Compact subsets of a Hausdorff space must be closed. The compact subsets of the real line. Continuous images of compact sets are compact. Quotient spaces. Continuous real-valued functions on a compact space are bounded and attain their bounds. The product of two compact spaces is compact. The compact subsets of Euclidean space. Sequential compactness. [3]

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0 Introduction

1 Metric spaces

1.1 Definitions

Proposition. If (X, d) is a metric space, (x_n) is a sequence in X such that $x_n \rightarrow x$, $x_n \rightarrow x'$, then $x = x'$.

Proof. For any $\varepsilon > 0$, we know that there exists N such that $d(x_n, x) < \varepsilon/2$ if $n > N$. Similarly, there exists some N' such that $d(x_n, x') < \varepsilon/2$ if $n > N'$.

Hence if $n > \max(N, N')$, then

$$\begin{aligned} 0 &\leq d(x, x') \\ &\leq d(x, x_n) + d(x_n, x') \\ &= d(x_n, x) + d(x_n, x') \\ &\leq \varepsilon. \end{aligned}$$

So $0 \leq d(x, x') \leq \varepsilon$ for all $\varepsilon > 0$. So $d(x, x') = 0$, and $x = x'$. □

1.2 Examples of metric spaces

1.3 Norms

Lemma. If $\|\cdot\|$ is a norm on V , then

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$

defines a metric on V .

Proof.

(i) $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\| \geq 0$ by the definition of the norm.

(ii) $d(\mathbf{v}, \mathbf{w}) = 0 \Leftrightarrow \|\mathbf{v} - \mathbf{w}\| = 0 \Leftrightarrow \mathbf{v} - \mathbf{w} = \mathbf{0} \Leftrightarrow \mathbf{v} = \mathbf{w}$.

(iii) $d(\mathbf{w}, \mathbf{v}) = \|\mathbf{w} - \mathbf{v}\| = \|(-1)(\mathbf{v} - \mathbf{w})\| = |-1|\|\mathbf{v} - \mathbf{w}\| = d(\mathbf{v}, \mathbf{w})$.

(iv) $d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w}) = \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\| \geq \|\mathbf{u} - \mathbf{w}\| = d(\mathbf{u}, \mathbf{w})$. □

Lemma. Let $f \in C[0, 1]$ satisfy $f(x) \geq 0$ for all $x \in [0, 1]$. If $f(x)$ is not constantly 0, then $\int_0^1 f(x) dx > 0$.

Proof. Pick $x_0 \in [0, 1]$ with $f(x_0) = a > 0$. Then since f is continuous, there is a δ such that $|f(x) - f(x_0)| < a/2$ if $|x - x_0| < \delta$. So $|f(x)| > a/2$ in this region.

Take

$$g(x) = \begin{cases} a/2 & |x - x_0| < \delta \\ 0 & \text{otherwise} \end{cases}$$

Then $f(x) \geq g(x)$ for all $x \in [0, 1]$. So

$$\int_0^1 f(x) dx \geq \int_0^1 g(x) dx = \frac{a}{2} \cdot (2\delta) > 0. \quad \square$$

Theorem (Cauchy-Schwarz inequality). If $\langle \cdot, \cdot \rangle$ is an inner product, then

$$\langle \mathbf{v}, \mathbf{w} \rangle^2 \leq \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle.$$

Proof. For any x , we have

$$\langle \mathbf{v} + x\mathbf{w}, \mathbf{v} + x\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle + 2x\langle \mathbf{v}, \mathbf{w} \rangle + x^2\langle \mathbf{w}, \mathbf{w} \rangle \geq 0.$$

Seen as a quadratic in x , since it is always non-negative, it can have at most one real root. So

$$(2\langle \mathbf{v}, \mathbf{w} \rangle)^2 - 4\langle \mathbf{v}, \mathbf{v} \rangle\langle \mathbf{w}, \mathbf{w} \rangle \leq 0.$$

So the result follows. \square

Lemma. If $\langle \cdot, \cdot \rangle$ is an inner product on V , then

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

is a norm.

Proof.

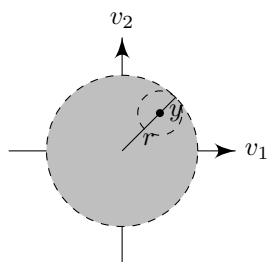
- (i) $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \geq 0$.
- (ii) $\|\mathbf{v}\| = 0 \Leftrightarrow \langle \mathbf{v}, \mathbf{v} \rangle = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$.
- (iii) $\|\lambda\mathbf{v}\| = \sqrt{\langle \lambda\mathbf{v}, \lambda\mathbf{v} \rangle} = \sqrt{\lambda^2\langle \mathbf{v}, \mathbf{v} \rangle} = |\lambda|\|\mathbf{v}\|$.
- (iv)

$$\begin{aligned} (\|\mathbf{v}\| + \|\mathbf{w}\|)^2 &= \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 \\ &\geq \langle \mathbf{v}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \\ &= \|\mathbf{v} + \mathbf{w}\|^2 \end{aligned} \quad \square$$

1.4 Open and closed subsets

Lemma. The open ball $B_r(x) \subseteq X$ is an open subset, and the closed ball $\bar{B}_r(x) \subseteq X$ is a closed subset.

Proof. Given $y \in B_r(x)$, we must find $\delta > 0$ with $B_\delta(y) \subseteq B_r(x)$.



Since $y \in B_r(x)$, we must have $a = d(y, x) < r$. Let $\delta = r - a > 0$. Then if $z \in B_\delta(y)$, then

$$d(z, x) \leq d(z, y) + d(y, x) < (r - a) + a = r.$$

So $z \in B_r(x)$. So $B_\delta(y) \subseteq B_r(x)$ as desired.

The second statement is equivalent to $X \setminus \bar{B}_r(x) = \{y \in X : d(y, x) > r\}$ is open. The proof is very similar. \square

Lemma. If U is an open neighbourhood of x and $x_n \rightarrow x$, then $\exists N$ such that $x_n \in U$ for all $n > N$.

Proof. Since U is open, there exists some $\delta > 0$ such that $B_\delta(x) \subseteq U$. Since $x_n \rightarrow x$, $\exists N$ such that $d(x_n, x) < \delta$ for all $n > N$. This implies that $x_n \in B_\delta(x)$ for all $n > N$. So $x_n \in U$ for all $n > N$. \square

Proposition. $C \subseteq X$ is a closed subset if and only if every limit point of C is an element of C .

Proof. (\Rightarrow) Suppose C is closed and $x_n \rightarrow x$, $x_n \in C$. We have to show that $x \in C$.

Since C is closed, $A = X \setminus C \subseteq X$ is open. Suppose the contrary that $x \notin C$. Then $x \in A$. Hence A is an open neighbourhood of x . Then by our previous lemma, we know that there is some N such that $x_n \in A$ for all $n \geq N$. So $x_N \in A$. But we know that $x_N \in C$ by assumption. This is a contradiction. So we must have $x \in C$.

(\Leftarrow) Suppose that C is not closed. We have to find a limit point not in C .

Since C is not closed, A is not open. So $\exists x \in A$ such that $B_\delta(x) \not\subseteq A$ for all $\delta > 0$. This means that $B_\delta(x) \cap C \neq \emptyset$ for all $\delta > 0$.

So pick $x_n \in B_{\frac{1}{n}}(x) \cap C$ for each $n > 0$. Then $x_n \in C$, $d(x_n, x) = \frac{1}{n} \rightarrow 0$. So $x_n \rightarrow x$. So x is a limit point of C which is not in C . \square

Proposition (Characterization of continuity). Let (X, d_x) and (Y, d_y) be metric spaces, and $f : X \rightarrow Y$. The following conditions are equivalent:

- (i) f is continuous
- (ii) If $x_n \rightarrow x$, then $f(x_n) \rightarrow f(x)$ (which is the definition of continuity)
- (iii) For any closed subset $C \subseteq Y$, $f^{-1}(C)$ is closed in X .
- (iv) For any open subset $U \subseteq Y$, $f^{-1}(U)$ is open in X .
- (v) For any $x \in X$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.
Alternatively, $d_x(x, z) < \delta \Rightarrow d_y(f(x), f(z)) < \varepsilon$.

Proof.

- 1 \Leftrightarrow 2: by definition
- 2 \Rightarrow 3: Suppose $C \subseteq Y$ is closed. We want to show that $f^{-1}(C)$ is closed. So let $x_n \rightarrow x$, where $x_n \in f^{-1}(C)$. We know that $f(x_n) \rightarrow f(x)$ by (2) and $f(x_n) \in C$. So $f(x)$ is a limit point of C . Since C is closed, $f(x) \in C$. So $x \in f^{-1}(C)$. So every limit point of $f^{-1}(C)$ is in $f^{-1}(C)$. So $f^{-1}(C)$ is closed.
- 3 \Rightarrow 4: If $U \subseteq Y$ is open, then $Y \setminus U$ is closed in Y . So $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed in X . So $f^{-1}(U) \subseteq X$ is open.
- 4 \Rightarrow 5: Given $x \in X, \varepsilon > 0$, $B_\varepsilon(f(x))$ is open in Y . By (4), we know $f^{-1}(B_\varepsilon(f(x))) = A$ is open in X . Since $x \in A$, $\exists \delta > 0$ with $B_\delta(x) \subseteq A$. So

$$f(B_\delta(x)) \subseteq f(A) = f(f^{-1}(B_\varepsilon(f(x)))) = B_\varepsilon(f(x))$$

- 5 \Rightarrow 2: Suppose $x_n \rightarrow x$. Given $\varepsilon > 0$, $\exists \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Since $x_n \rightarrow x$, $\exists N$ such that $x_n \in B_\delta(x)$ for all $n > N$. Then $f(x_n) \in f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ for all $n > N$. So $f(x_n) \rightarrow f(x)$. \square

Lemma.

- (i) \emptyset and X are open subsets of X .
- (ii) Suppose $V_\alpha \subseteq X$ is open for all $\alpha \in A$. Then $U = \bigcup_{\alpha \in A} V_\alpha$ is open in X .
- (iii) If $V_1, \dots, V_n \subseteq X$ are open, then so is $V = \bigcap_{i=1}^n V_i$.

Proof.

- (i) \emptyset satisfies the definition of an open subset vacuously. X is open since for any x , $B_1(x) \subseteq X$.
- (ii) If $x \in U$, then $x \in V_\alpha$ for some α . Since V_α is open, there exists $\delta > 0$ such that $B_\delta(x) \subseteq V_\alpha$. So $B_\delta(x) \subseteq \bigcup_{\alpha \in A} V_\alpha = U$. So U is open.
- (iii) If $x \in V$, then $x \in V_i$ for all $i = 1, \dots, n$. So $\exists \delta_i > 0$ with $B_{\delta_i}(x) \subseteq V_i$. Take $\delta = \min\{\delta_1, \dots, \delta_n\}$. So $B_\delta(x) \subseteq V_i$ for all i . So $B_\delta(x) \subseteq V$. So V is open. \square

2 Topological spaces

2.1 Definitions

Lemma. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then so is $g \circ f : X \rightarrow Z$.

Proof. If $U \subseteq Z$ is open, g is continuous, then $g^{-1}(U)$ is open in Y . Since f is also continuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in X . \square

Lemma. Homeomorphism is an equivalence relation.

Proof.

- (i) The identity map $I_X : X \rightarrow X$ is always a homeomorphism. So $X \simeq X$.
- (ii) If $f : X \rightarrow Y$ is a homeomorphism, then so is $f^{-1} : Y \rightarrow X$. So $X \simeq Y \Rightarrow Y \simeq X$.
- (iii) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are homeomorphisms, then $g \circ f : X \rightarrow Z$ is a homeomorphism. So $X \simeq Y$ and $Y \simeq Z$ implies $X \simeq Z$. \square

2.2 Sequences

Lemma. If X is Hausdorff, x_n is a sequence in X with $x_n \rightarrow x$ and $x_n \rightarrow x'$, then $x = x'$, i.e. limits are unique.

Proof. Suppose the contrary that $x \neq x'$. Then by definition of Hausdorff, there exist open neighbourhoods U, U' of x, x' respectively with $U \cap U' = \emptyset$.

Since $x_n \rightarrow x$ and U is a neighbourhood of x , by definition, there is some N such that whenever $n > N$, we have $x_n \in U$. Similarly, since $x_n \rightarrow x'$, there is some N' such that whenever $n > N'$, we have $x_n \in U'$.

This means that whenever $n > \max(N, N')$, we have $x_n \in U$ and $x_n \in U'$. So $x_n \in U \cap U'$. This contradicts the fact that $U \cap U' = \emptyset$.

Hence we must have $x = x'$. \square

2.3 Closed sets

Lemma.

- (i) If C_α is a closed subset of X for all $\alpha \in A$, then $\bigcap_{\alpha \in A} C_\alpha$ is closed in X .
- (ii) If C_1, \dots, C_n are closed in X , then so is $\bigcup_{i=1}^n C_i$.

Proof.

- (i) Since C_α is closed in X , $X \setminus C_\alpha$ is open in X . So $\bigcup_{\alpha \in A} (X \setminus C_\alpha) = X \setminus \bigcap_{\alpha \in A} C_\alpha$ is open. So $\bigcap_{\alpha \in A} C_\alpha$ is closed.
- (ii) If C_i is closed in X , then $X \setminus C_i$ is open. So $\bigcap_{i=1}^n (X \setminus C_i) = X \setminus \bigcup_{i=1}^n C_i$ is open. So $\bigcup_{i=1}^n C_i$ is closed. \square

Corollary. If X is Hausdorff and $x \in X$, then $\{x\}$ is closed in X .

Proof. For all $y \in X$, there exist open subsets U_y, V_y with $y \in U_y, x \in V_y, U_y \cap V_y = \emptyset$.

Let $C_y = X \setminus U_y$. Then C_y is closed, $y \notin C_y, x \in C_y$. So $\{x\} = \bigcap_{y \neq x} C_y$ is closed since it is an intersection of closed subsets. \square

2.4 Closure and interior

2.4.1 Closure

Proposition. \bar{A} is the smallest closed subset of X which contains A .

Proof. Let $K \subseteq X$ be a closed set containing A . Then $K \in \mathcal{C}_A$. So $\bar{A} = \bigcap_{C \in \mathcal{C}_A} C \subseteq K$. So $\bar{A} \subseteq K$. \square

Lemma. If $C \subseteq X$ is closed, then $L(C) = C$.

Proof. Exactly the same as that for metric spaces. We will also prove a more general result very soon that implies this. \square

Proposition. $L(A) \subseteq \bar{A}$.

Proof. If $A \subseteq C$, then $L(A) \subseteq L(C)$. If C is closed, then $L(C) = C$. So $C \in \mathcal{C}_A \Rightarrow L(A) \subseteq C$. So $L(A) \subseteq \bigcap_{C \in \mathcal{C}_A} C = \bar{A}$. \square

Corollary. Given a subset $A \subseteq X$, if we can find some closed C such that $A \subseteq C \subseteq L(A)$, then we in fact have $C = \bar{A}$.

Proof. $C \subseteq L(A) \subseteq \bar{A} \subseteq C$, where the last step is since \bar{A} is the smallest closed set containing A . So $C = L(A) = \bar{A}$. \square

2.4.2 Interior

Proposition. $\text{Int}(A)$ is the largest open subset of X contained in A .

Proposition. $X \setminus \text{Int}(A) = \overline{X \setminus A}$

Proof. $U \subseteq A \Leftrightarrow (X \setminus U) \supseteq (X \setminus A)$. Also, U open in $X \Leftrightarrow X \setminus U$ is closed in X .

So the complement of the largest open subset of X contained in A will be the smallest closed subset containing $X \setminus A$. \square

2.5 New topologies from old

2.5.1 Subspace topology

Proposition. The subspace topology is a topology.

Proof.

- (i) Since \emptyset is open in X , $\emptyset = Y \cap \emptyset$ is open in Y .
Since X is open in X , $Y = Y \cap X$ is open in Y .
- (ii) If V_α is open in Y , then $V_\alpha = Y \cap U_\alpha$ for some U_α open in X . Then

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A} (Y \cap U_\alpha) = Y \cap \left(\bigcup_{\alpha \in U} U_\alpha \right).$$

Since $\bigcup U_\alpha$ is open in X , so $\bigcup V_\alpha$ is open in Y .

(iii) If V_i is open in Y , then $V_i = Y \cap U_i$ for some open $U_i \subseteq X$. Then

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n (Y \cap U_i) = Y \cap \left(\bigcap_{i=1}^n U_i \right).$$

Since $\bigcap U_i$ is open, $\bigcap V_i$ is open. □

Proposition. If Y has the subspace topology, $f : Z \rightarrow Y$ is continuous iff $\iota \circ f : Z \rightarrow X$ is continuous.

Proof. (\Rightarrow) If $U \subseteq X$ is open, then $\iota^{-1}(U) = Y \cap U$ is open in Y . So ι is continuous. So if f is continuous, so is $\iota \circ f$.

(\Leftarrow) Suppose we know that $\iota \circ f$ is continuous. Given $V \subseteq Y$ is open, we know that $V = Y \cap U = \iota^{-1}(U)$. So $f^{-1}(V) = f^{-1}(\iota^{-1}(U)) = (\iota \circ f)^{-1}(U)$ is open since $\iota \circ f$ is continuous. So f is continuous. □

2.5.2 Product topology

2.5.3 Quotient topology

3 Connectivity

3.1 Connectivity

Proposition. X is disconnected iff there exists a continuous surjective $f : X \rightarrow \{0, 1\}$ with the discrete topology.

Alternatively, X is connected iff any continuous map $f : X \rightarrow \{0, 1\}$ is constant.

Proof. (\Rightarrow) If A and B disconnect X , define

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

Then $f^{-1}(\emptyset)$, $f^{-1}(\{0, 1\}) = X$, $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$ are all open. So f is continuous. Also, since A, B are non-empty, f is surjective.

(\Leftarrow) Given $f : X \rightarrow \{0, 1\}$ surjective and continuous, define $A = f^{-1}(\{0\})$, $B = f^{-1}(\{1\})$. Then A and B disconnect X . \square

Theorem. $[0, 1]$ is connected.

Proof. Suppose A and B disconnect $[0, 1]$. wlog, assume $1 \in B$. Since A is non-empty, $\alpha = \sup A$ exists. Then either

- $\alpha \in A$. Then $\alpha < 1$, since $1 \in B$. Since A is open, $\exists \varepsilon > 0$ with $B_\varepsilon(\alpha) \subseteq A$. So $\alpha + \frac{\varepsilon}{2} \in A$, contradicting supremality of α ; or
- $\alpha \notin A$. Then $\alpha \in B$. Since B is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(\alpha) \subseteq B$. Then $a \leq \alpha - \varepsilon$ for all $a \in A$. This contradicts α being the *least* upper bound of A .

Either option gives a contradiction. So A and B cannot exist and $[0, 1]$ is connected. \square

Proposition. If $f : X \rightarrow Y$ is continuous and X is connected, then $\text{im } f$ is also connected.

Proof. Suppose A and B disconnect $\text{im } f$. We will show that $f^{-1}(A)$ and $f^{-1}(B)$ disconnect X .

Since $A, B \subseteq \text{im } f$ are open, we know that $A = \text{im } f \cap A'$ and $B = \text{im } f \cap B'$ for some A', B' open in Y . Then $f^{-1}(A) = f^{-1}(A')$ and $f^{-1}(B) = f^{-1}(B')$ are open in X .

Since A, B are non-empty, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty. Also, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$. Finally, $A \cup B = \text{im } f$. So $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = X$.

So $f^{-1}(A)$ and $f^{-1}(B)$ disconnect X , contradicting our hypothesis. So $\text{im } f$ is connected. \square

Theorem (Intermediate value theorem). Suppose $f : X \rightarrow \mathbb{R}$ is continuous and X is connected. If $\exists x_0, x_1$ such that $f(x_0) < 0 < f(x_1)$, then $\exists x \in X$ with $f(x) = 0$.

Proof. Suppose no such x exists. Then $0 \notin \text{im } f$ while $0 > f(x_0) \in \text{im } f$, $0 < f(x_1) \in \text{im } f$. Then $\text{im } f$ is disconnected (from our previous example), contradicting X being connected. \square

Corollary. If $f : [0, 1] \rightarrow \mathbb{R}$ is continuous with $f(0) < 0 < f(1)$, then $\exists x \in [0, 1]$ with $f(x) = 0$.

3.2 Path connectivity

Proposition. If X is path connected, then X is connected.

Proof. Let X be path connected, and let $f : X \rightarrow \{0, 1\}$ be a continuous function. We want to show that f is constant.

Let $x, y \in X$. By path connectedness, there is a map $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Composing with f gives a map $f \circ \gamma : [0, 1] \rightarrow \{0, 1\}$. Since $[0, 1]$ is connected, this must be constant. In particular, $f(\gamma(0)) = f(\gamma(1))$, i.e. $f(x) = f(y)$. Since x, y were arbitrary, we know f is constant. \square

Lemma. Suppose $f : X \rightarrow Y$ is a homeomorphism and $A \subseteq X$, then $f|_A : A \rightarrow f(A)$ is a homeomorphism.

Proof. Since f is a bijection, $f|_A$ is a bijection. If $U \subseteq f(A)$ is open, then $U = f(A) \cap U'$ for some U' open in Y . So $f|_A^{-1}(U) = f^{-1}(U') \cap A$ is open in A . So $f|_A$ is continuous. Similarly, we can show that $(f|_A)^{-1}$ is continuous. \square

3.2.1 Higher connectivity*

3.3 Components

3.3.1 Path components

Lemma. Define $x \sim y$ if there is a path from x to y in X . Then \sim is an equivalence relation.

Proof.

- (i) For any $x \in X$, let $\gamma_x : [0, 1] \rightarrow X$ be $\gamma(t) = x$, the constant path. Then this is a path from x to x . So $x \sim x$.
- (ii) If $\gamma : [0, 1] \rightarrow X$ is a path from x to y , then $\bar{\gamma} : [0, 1] \rightarrow X$ by $t \mapsto \gamma(1 - t)$ is a path from y to x . So $x \sim y \Rightarrow y \sim x$.
- (iii) If γ_1 is a path from x to y and γ_2 is a path from y to z , then $\gamma_2 * \gamma_1$ defined by

$$t \mapsto \begin{cases} \gamma_1(2t) & t \in [0, 1/2] \\ \gamma_2(2t - 1) & t \in [1/2, 1] \end{cases}$$

is a path from x to z . So $x \sim y, y \sim z \Rightarrow x \sim z$. \square

3.3.2 Connected components

Proposition. Suppose $Y_\alpha \subseteq X$ is connected for all $\alpha \in T$ and that $\bigcap_{\alpha \in T} Y_\alpha \neq \emptyset$. Then $Y = \bigcup_{\alpha \in T} Y_\alpha$ is connected.

Proof. Suppose the contrary that A and B disconnect Y . Then A and B are open in Y . So $A = Y \cap A'$ and $B = Y \cap B'$, where A', B' are open in X . For any fixed α , let

$$A_\alpha = Y_\alpha \cap A = Y_\alpha \cap A', \quad B_\alpha = Y_\alpha \cap B = Y_\alpha \cap B'.$$

Then they are open in Y_α . Since $Y = A \cup B$, we have

$$Y_\alpha = Y \cap Y_\alpha = (A \cup B) \cap Y_\alpha = A_\alpha \cup B_\alpha.$$

Since $A \cap B = \emptyset$, we have

$$A_\alpha \cap B_\alpha = Y_\alpha \cap (A \cap B) = \emptyset.$$

So A_α, B_α are disjoint. So Y_α is connected but is the disjoint union of open subsets A_α, B_α .

By definition of connectivity, this can only happen if $A_\alpha = \emptyset$ or $B_\alpha = \emptyset$.

However, by assumption, $\bigcap_{\alpha \in T} Y_\alpha \neq \emptyset$. So pick $y \in \bigcap_{\alpha \in T} Y_\alpha$. Since $y \in Y$, either $y \in A$ or $y \in B$. wlog, assume $y \in A$. Then $y \in Y_\alpha$ for all α implies that $y \in A_\alpha$ for all α . So A_α is non-empty for all α . So B_α is empty for all α . So $B = \emptyset$.

So A and B did not disconnect Y after all. Contradiction. \square

Lemma. If $y \in C(x)$, then $C(y) = C(x)$.

Proof. Since $y \in C(x)$ and $C(x)$ is connected, $C(x) \subseteq C(y)$. So $x \in C(y)$. Then $C(y) \subseteq C(x)$. So $C(x) = C(y)$. \square

Proposition. If $U \subseteq \mathbb{R}^n$ is open and connected, then it is path-connected.

Proof. Let A be a path component of U . We first show that A is open.

Let $a \in A$. Since U is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(a) \subseteq U$. We know that $B_\varepsilon(a) \simeq \text{Int}(D^n)$ is path-connected (e.g. use line segments connecting the points). Since A is a path component and $a \in A$, we must have $B_\varepsilon(a) \subseteq A$. So A is an open subset of U .

Now suppose $b \in U \setminus A$. Then since U is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(b) \subseteq U$. Since $B_\varepsilon(b)$ is path-connected, so if $B_\varepsilon(b) \cap A \neq \emptyset$, then $B_\varepsilon(b) \subseteq A$. But this implies $b \in A$, which is a contradiction. So $B_\varepsilon(b) \cap A = \emptyset$. So $B_\varepsilon(b) \subseteq U \setminus A$. Then $U \setminus A$ is open.

So $A, U \setminus A$ are disjoint open subsets of U . Since U is connected, we must have $U \setminus A$ empty (since A is not). So $U = A$ is path-connected. \square

4 Compactness

4.1 Compactness

Theorem. $[0, 1]$ is compact.

Proof. Suppose \mathcal{V} is an open cover of $[0, 1]$. Let

$$A = \{a \in [0, 1] : [0, a] \text{ has a finite subcover of } \mathcal{V}\}.$$

First show that A is non-empty. Since \mathcal{V} covers $[0, 1]$, in particular, there is some V_0 that contains 0. So $\{0\}$ has a finite subcover V_0 . So $0 \in A$.

Next we note that by definition, if $0 \leq b \leq a$ and $a \in A$, then $b \in A$.

Now let $\alpha = \sup A$. Suppose $\alpha < 1$. Then $\alpha \in [0, 1]$.

Since \mathcal{V} covers X , let $\alpha \in V_\alpha$. Since V_α is open, there is some ε such that $B_\varepsilon(\alpha) \subseteq V_\alpha$. By definition of α , we must have $\alpha - \varepsilon/2 \in A$. So $[0, \alpha - \varepsilon/2]$ has a finite subcover. Add V_α to that subcover to get a finite subcover of $[0, \alpha + \varepsilon/2]$. Contradiction (technically, it will be a finite subcover of $[0, \eta]$ for $\eta = \min(\alpha + \varepsilon/2, 1)$, in case $\alpha + \varepsilon/2$ gets too large).

So we must have $\alpha = \sup A = 1$.

Now we argue as before: $\exists V_1 \in \mathcal{V}$ such that $1 \in V_1$ and $\exists \varepsilon > 0$ with $(1 - \varepsilon, 1] \subseteq V_1$. Since $1 - \varepsilon \in A$, there exists a finite $\mathcal{V}' \subseteq \mathcal{V}$ which covers $[0, 1 - \varepsilon/2]$. Then $\mathcal{W} = \mathcal{V}' \cup \{V_1\}$ is a finite subcover of \mathcal{V} . \square

Proposition. If X is compact and C is a closed subset of X , then C is also compact.

Proof. To prove this, given an open cover of C , we need to find a finite subcover. To do so, we need to first convert it into an open cover of X . We can do so by adding $X \setminus C$, which is open since C is closed. Then since X is compact, we can find a finite subcover of this, which we can convert back to a finite subcover of C .

Formally, suppose \mathcal{V} is an open cover of C . Say $\mathcal{V} = \{V_\alpha : \alpha \in T\}$. For each α , since V_α is open in C , $V_\alpha = C \cap V'_\alpha$ for some V'_α open in X . Also, since $\bigcup_{\alpha \in T} V_\alpha = C$, we have $\bigcup_{\alpha \in T} V'_\alpha \supseteq C$.

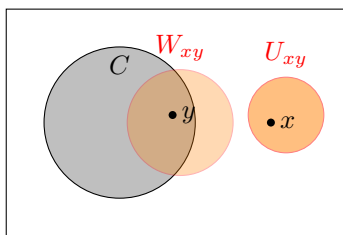
Since C is closed, $U = X \setminus C$ is open in X . So $\mathcal{W} = \{V'_\alpha : \alpha \in T\} \cup \{U\}$ is an open cover of X . Since X is compact, \mathcal{W} has a finite subcover $\mathcal{W}' = \{V'_{\alpha_1}, \dots, V'_{\alpha_n}, U\}$ (U may or may not be in there, but it doesn't matter). Now $U \cap C = \emptyset$. So $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ is a finite subcover of C . \square

Proposition. Let X be a Hausdorff space. If $C \subseteq X$ is compact, then C is closed in X .

Proof. Let $U = X \setminus C$. We will show that U is open.

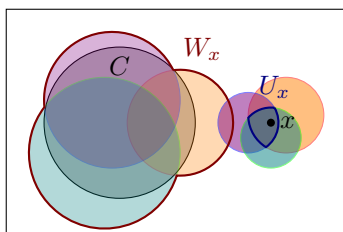
For any x , we will find a U_x such that $U_x \subseteq U$ and $x \in U_x$. Then $U = \bigcup_{x \in U} U_x$ will be open since it is a union of open sets.

To construct U_x , fix $x \in U$. Since X is Hausdorff, for each $y \in C$, $\exists U_{xy}, W_{xy}$ open neighbourhoods of x and y respectively with $U_{xy} \cap W_{xy} = \emptyset$.



Then $\mathcal{W} = \{W_{xy} \cap C : y \in C\}$ is an open cover of C . Since C is compact, there exists a finite subcover $\mathcal{W}' = \{W_{xy_1} \cap C, \dots, W_{xy_n} \cap C\}$.

Let $U_x = \bigcap_{i=1}^n U_{xy_i}$. Then U_x is open since it is a finite intersection of open sets. To show $U_x \subseteq U$, note that $W_x = \bigcup_{i=1}^n W_{xy_i} \supseteq C$ since $\{W_{xy_i} \cap C\}$ is an open cover. We also have $W_x \cap U_x = \emptyset$. So $U_x \subseteq U$. So done.



□

Proposition. A compact metric space (X, d) is bounded.

Proof. Pick $x \in X$. Then $V = \{B_r(x) : r \in \mathbb{R}^+\}$ is an open cover of X . Since X is compact, there is a finite subcover $\{B_{r_1}(x), \dots, B_{r_n}(x)\}$.

Let $R = \max\{r_1, \dots, r_n\}$. Then $d(x, y) < R$ for all $y \in X$. So for all $y, z \in X$,

$$d(y, z) \leq d(y, x) + d(x, z) < 2R$$

So X is bounded. □

Theorem (Heine-Borel). $C \subseteq \mathbb{R}$ is compact iff C is closed and bounded.

Proof. Since \mathbb{R} is a metric space (hence Hausdorff), C is also a metric space.

So if C is compact, C is closed in \mathbb{R} , and C is bounded, by our previous two propositions.

Conversely, if C is closed and bounded, then $C \subseteq [-N, N]$ for some $N \in \mathbb{R}$. Since $[-N, N] \simeq [0, 1]$ is compact, and $C = C \cap [-N, N]$ is closed in $[-N, N]$, C is compact. □

Corollary. If $A \subseteq \mathbb{R}$ is compact, $\exists \alpha \in A$ such that $\alpha \geq a$ for all $a \in A$.

Proof. Since A is compact, it is bounded. Let $\alpha = \sup A$. Then by definition, $\alpha \geq a$ for all $a \in A$. So it is enough to show that $\alpha \in A$.

Suppose $\alpha \notin A$. Then $\alpha \in \mathbb{R} \setminus A$. Since A is compact, it is closed in \mathbb{R} . So $\mathbb{R} \setminus A$ is open. So $\exists \varepsilon > 0$ such that $B_\varepsilon(\alpha) \subseteq \mathbb{R} \setminus A$, which implies that $a \leq \alpha - \varepsilon$ for all $a \in A$. This contradicts the assumption that $\alpha = \sup A$. So we can conclude $\alpha \in A$. □

Proposition. If $f : X \rightarrow Y$ is continuous and X is compact, then $\text{im } f \subseteq Y$ is also compact.

Proof. Suppose $\mathcal{V} = \{V_\alpha : \alpha \in T\}$ is an open cover of $\text{im } f$. Since V_α is open in $\text{im } f$, we have $V_\alpha = \text{im } f \cap V'_\alpha$, where V'_α is open in Y . Then

$$W_\alpha = f^{-1}(V_\alpha) = f^{-1}(V'_\alpha)$$

is open in X . If $x \in X$ then $f(x)$ is in V_α for some α , so $x \in W_\alpha$. Thus $\mathcal{W} = \{W_\alpha : \alpha \in T\}$ is an open cover of X .

Since X is compact, there's a finite subcover $\{W_{\alpha_1}, \dots, W_{\alpha_n}\}$ of \mathcal{W} .

Since $V_\alpha \subseteq \text{im } f$, $f(W_\alpha) = f(f^{-1}(V_\alpha)) = V_\alpha$. So

$$\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$$

is a finite subcover of \mathcal{V} . □

Theorem (Maximum value theorem). If $f : X \rightarrow \mathbb{R}$ is continuous and X is compact, then $\exists x \in X$ such that $f(x) \geq f(y)$ for all $y \in X$.

Proof. Since X is compact, $\text{im } f$ is compact. Let $\alpha = \max\{\text{im } f\}$. Then $\alpha \in \text{im } f$. So $\exists x \in X$ with $f(x) = \alpha$. Then by definition $f(x) \geq f(y)$ for all $y \in X$. □

Corollary. If $f : [0, 1] \rightarrow \mathbb{R}$ is continuous, then $\exists x \in [0, 1]$ such that $f(x) \geq f(y)$ for all $y \in [0, 1]$

Proof. $[0, 1]$ is compact. □

4.2 Products and quotients

4.2.1 Products

Theorem. If X and Y are compact, then so is $X \times Y$.

Proof. First consider the special type of open cover \mathcal{V} of $X \times Y$ such that every $U \in \mathcal{V}$ has the form $U = V \times W$, where $V \subseteq X$ and $W \subseteq Y$ are open.

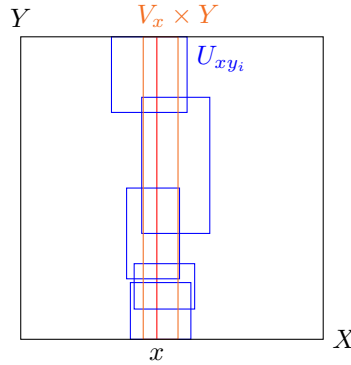
For every $(x, y) \in X \times Y$, there is $U_{xy} \in \mathcal{V}$ with $(x, y) \in U_{xy}$. Write

$$U_{xy} = V_{xy} \times W_{xy},$$

where $V_{xy} \subseteq X$, $W_{xy} \subseteq Y$ are open, $x \in V_{xy}$, $y \in W_{xy}$.

Fix $x \in X$. Then $\mathcal{W}_x = \{W_{xy} : y \in Y\}$ is an open cover of Y . Since Y is compact, there is a finite subcover $\{W_{xy_1}, \dots, W_{xy_n}\}$.

Then $V_x = \bigcap_{i=1}^n V_{xy_i}$ is a finite intersection of open sets. So V_x is open in X . Moreover, $\mathcal{V}_x = \{U_{xy_1}, \dots, U_{xy_n}\}$ covers $V_x \times Y$.



Now $\mathcal{O} = \{V_x : x \in X\}$ is an open cover of X . Since X is compact, there is a finite subcover $\{V_{x_1}, \dots, V_{x_m}\}$. Then $\mathcal{V}' = \bigcup_{i=1}^m \mathcal{V}_{x_i}$ is a finite subset of \mathcal{V} , which covers all of $X \times Y$.

In the general case, suppose \mathcal{V} is an open cover of $X \times Y$. For each $(x, y) \in X \times Y$, $\exists U_{xy} \in \mathcal{V}$ with $(x, y) \in U_{xy}$. Since U_{xy} is open, $\exists V_{xy} \in X, W_{xy} \subseteq Y$ open with $V_{xy} \times W_{xy} \subseteq U_{xy}$ and $x \in V_{xy}, y \in W_{xy}$.

Then $\mathcal{Q} = \{V_{xy} \times W_{xy} : (x, y) \in (X, Y)\}$ is an open cover of $X \times Y$ of the type we already considered above. So it has a finite subcover $\{V_{x_1 y_1} \times W_{x_1 y_1}, \dots, V_{x_n y_n} \times W_{x_n y_n}\}$. Now $V_{x_i y_i} \times W_{x_i y_i} \subseteq U_{x_i y_i}$. So $\{U_{x_1 y_1}, \dots, U_{x_n y_n}\}$ is a finite subcover of $X \times Y$. \square

Corollary (Heine-Borel in \mathbb{R}^n). $C \subseteq \mathbb{R}^n$ is compact iff C is closed and bounded.

Proof. If C is bounded, $C \subseteq [-N, N]^n$ for some $N \in \mathbb{R}$, which is compact. The rest of the proof is exactly the same as for $n = 1$. \square

4.2.2 Quotients

Proposition. Suppose $f : X \rightarrow Y$ is a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. We show that f^{-1} is continuous. To do this, it suffices to show $(f^{-1})^{-1}(C)$ is closed in X whenever C is closed in Y . By hypothesis, f is a bijection. So $(f^{-1})^{-1}(C) = f(C)$.

Supposed C is closed in Y . Since X is compact, C is compact. Since f is continuous, $f(C) = (\text{im } f|_C)$ is compact. Since Y is Hausdorff and $f(C) \subseteq Y$ is compact, $f(C)$ is closed. \square

Corollary. Suppose $f : X/\sim \rightarrow Y$ is a bijection, X is compact, Y is Hausdorff, and $f \circ \pi$ is continuous, then f is a homeomorphism.

Proof. Since X is compact and $\pi : X \rightarrow X/\sim$ is continuous, $\text{im } \pi \subseteq X/\sim$ is compact. Since $f \circ \pi$ is continuous, f is continuous. So we can apply the proposition. \square

4.3 Sequential compactness

Lemma. Let (x_n) be a sequence in a metric space (X, d) and $x \in X$. Then (x_n) has a subsequence converging to x iff for every $\varepsilon > 0$, $x_n \in B_\varepsilon(x)$ for infinitely many n (*).

Proof. If $(x_{n_i}) \rightarrow x$, then for every ε , we can find I such that $i > I$ implies $x_{n_i} \in B_\varepsilon(x)$ by definition of convergence. So (*) holds.

Now suppose (*) holds. We will construct a sequence $x_{n_i} \rightarrow x$ inductively. Take $n_0 = 0$. Suppose we have defined $x_{n_0}, \dots, x_{n_{i-1}}$.

By hypothesis, $x_n \in B_{1/i}(x)$ for infinitely many n . Take n_i to be smallest such n with $n_i > n_{i-1}$.

Then $d(x_{n_i}, x) < \frac{1}{i}$ implies that $x_{n_i} \rightarrow x$. \square

Theorem. If (X, d) is a compact metric space, then X is sequentially compact.

Proof. Suppose x_n is a sequence in X with no convergent subsequence. Then for any $y \in X$, there is no subsequence converging to y . By lemma, there exists $\varepsilon > 0$ such that $x_n \in B_\varepsilon(y)$ for only finitely many n .

Let $U_y = B_\varepsilon(y)$. Now $\mathcal{V} = \{U_y : y \in X\}$ is an open cover of X . Since X is compact, there is a finite subcover $\{U_{y_1}, \dots, U_{y_m}\}$. Then $x_n \in \bigcup_{i=1}^m U_{y_i} = X$ for only finitely many n . This is nonsense, since $x_n \in X$ for all n !

So x_n must have a convergent subsequence. □

4.4 Completeness

Proposition. If X is a compact metric space, then X is complete.

Proof. Let x_n be a Cauchy sequence in X . Since X is sequentially compact, there is a convergent subsequence $x_{n_i} \rightarrow x$. We will show that $x_n \rightarrow x$.

Given $\varepsilon > 0$, pick N such that $d(x_n, x_m) < \varepsilon/2$ for $n, m \geq N$. Pick I such that $n_I \geq N$ and $d(x_{n_i}, x) < \varepsilon/2$ for all $i > I$. Then for $n \geq n_I$, $d(x_n, x) \leq d(x_n, x_{n_I}) + d(x_{n_I}, x) < \varepsilon$. So $x_n \rightarrow x$. □

Corollary. \mathbb{R}^n is complete.

Proof. If $(x_n) \subseteq \mathbb{R}^n$ is Cauchy, then $(x_n) \subseteq \bar{B}_R(0)$ for some R , and $\bar{B}_R(0)$ is compact. So it converges. □