Part IB — Metric and Topological Spaces
Theorems with proof

Based on lectures by J. Rasmussen
Notes taken by Dexter Chua

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Metrics
Definition and examples. Limits and continuity. Open sets and neighbourhoods. Characterizing limits and continuity using neighbourhoods and open sets. [3]

Topology
Definition of a topology. Metric topologies. Further examples. Neighbourhoods, closed sets, convergence and continuity. Hausdorff spaces. Homeomorphisms. Topological and non-topological properties. Completeness. Subspace, quotient and product topologies. [3]

Connectedness
Definition using open sets and integer-valued functions. Examples, including intervals. Components. The continuous image of a connected space is connected. Path-connectedness. Path-connected spaces are connected but not conversely. Connected open sets in Euclidean space are path-connected. [3]

Compactness
Definition using open covers. Examples: finite sets and [0, 1]. Closed subsets of compact spaces are compact. Compact subsets of a Hausdorff space must be closed. The compact subsets of the real line. Continuous images of compact sets are compact. Quotient spaces. Continuous real-valued functions on a compact space are bounded and attain their bounds. The product of two compact spaces is compact. The compact subsets of Euclidean space. Sequential compactness. [3]

Contents

0 Introduction

1 Metric spaces

1.1 Definitions

Proposition. If (X, d) is a metric space, (x_n) is a sequence in X such that $x_n \to x$, $x_n \to x'$, then $x = x'$.

Proof. For any $\varepsilon > 0$, we know that there exists N such that $d(x_n, x) < \varepsilon/2$ if $n > N$. Similarly, there exists some N' such that $d(x_n, x') < \varepsilon/2$ if $n > N'$.

Hence if $n > \max(N, N')$, then

$$\begin{aligned} 0 &\leq d(x, x') \\ &\leq d(x, x_n) + d(x_n, x') \\ &= d(x_n, x) + d(x_n, x') \\ &\leq \varepsilon. \end{aligned}$$

So $0 \leq d(x, x') \leq \varepsilon$ for all $\varepsilon > 0$. So $d(x, x') = 0$, and $x = x'$. □

1.2 Examples of metric spaces

1.3 Norms

Lemma. If $\|\cdot\|$ is a norm on V, then

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$

defines a metric on V.

Proof.

(i) $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\| \geq 0$ by the definition of the norm.

(ii) $d(\mathbf{v}, \mathbf{w}) = 0 \Leftrightarrow \|\mathbf{v} - \mathbf{w}\| = 0 \Leftrightarrow \mathbf{v} - \mathbf{w} = \mathbf{0} \Leftrightarrow \mathbf{v} = \mathbf{w}$.

(iii) $d(\mathbf{w}, \mathbf{v}) = \|\mathbf{w} - \mathbf{v}\| = \|(-1)(\mathbf{v} - \mathbf{w})\| = |-1|\|\mathbf{v} - \mathbf{w}\| = d(\mathbf{v}, \mathbf{w})$.

(iv) $d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w}) = \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\| \geq \|\mathbf{u} - \mathbf{w}\| = d(\mathbf{u}, \mathbf{w})$. □

Lemma. Let $f \in C[0, 1]$ satisfy $f(x) \geq 0$ for all $x \in [0, 1]$. If $f(x)$ is not constantly 0, then $\int_0^1 f(x)\;\mathrm{d}x > 0$.

Proof. Pick $x_0 \in [0, 1]$ with $f(x_0) = a > 0$. Then since f is continuous, there is a δ such that $|f(x) - f(x_0)| < a/2$ if $|x - x_0| < \delta$. So $|f(x)| > a/2$ in this region.

Take

$$g(x) = \begin{cases} a/2 & |x - x_0| < \delta \\ 0 & \text{otherwise} \end{cases}$$

Then $f(x) \geq g(x)$ for all $x \in [0, 1]$. So

$$\int_0^1 f(x)\;\mathrm{d}x \geq \int_0^1 g(x)\;\mathrm{d}x = \frac{a}{2} \cdot (2\delta) > 0.$$ □

Theorem (Cauchy-Schwarz inequality)**.** If $\langle\,\cdot\,,\,\cdot\,\rangle$ is an inner product, then

$$\langle \mathbf{v}, \mathbf{w}\rangle^2 \leq \langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{w}, \mathbf{w}\rangle.$$

Proof. For any x, we have

$$\langle \mathbf{v} + x\mathbf{w}, \mathbf{v} + x\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle + 2x\langle \mathbf{v}, \mathbf{w}\rangle + x^2\langle \mathbf{w}, \mathbf{w}\rangle \geq 0.$$

Seen as a quadratic in x, since it is always non-negative, it can have at most one real root. So

$$(2\langle \mathbf{v}, \mathbf{w}\rangle)^2 - 4\langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{w}, \mathbf{w}\rangle \leq 0.$$

So the result follows. □

Lemma. If $\langle\,\cdot\,,\,\cdot\,\rangle$ is an inner product on V, then

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$

is a norm.

Proof.

(i) $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} \geq 0$.

(ii) $\|\mathbf{v}\| = 0 \Leftrightarrow \langle \mathbf{v}, \mathbf{v}\rangle = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$.

(iii) $\|\lambda\mathbf{v}\| = \sqrt{\langle \lambda\mathbf{v}, \lambda\mathbf{v}\rangle} = \sqrt{\lambda^2\langle \mathbf{v}, \mathbf{v}\rangle} = |\lambda|\|\mathbf{v}\|$.

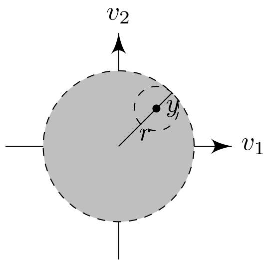
(iv)

$$\begin{aligned}(\|\mathbf{v}\| + \|\mathbf{w}\|)^2 &= \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2\\ &\geq \langle \mathbf{v}, \mathbf{v}\rangle + 2\langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle\\ &= \|\mathbf{v} + \mathbf{w}\|^2\end{aligned}$$ □

1.4 Open and closed subsets

Lemma. The open ball $B_r(x) \subseteq X$ is an open subset, and the closed ball $\bar{B}_r(x) \subseteq X$ is a closed subset.

Proof. Given $y \in B_r(x)$, we must find $\delta > 0$ with $B_\delta(y) \subseteq B_r(x)$.



Since $y \in B_r(x)$, we must have $a = d(y, x) < r$. Let $\delta = r - a > 0$. Then if $z \in B_\delta(y)$, then

$$d(z, x) \leq d(z, y) + d(y, x) < (r - a) + a = r.$$

So $z \in B_r(x)$. So $B_\delta(y) \subseteq B_r(x)$ as desired.

The second statement is equivalent to $X \setminus \bar{B}_r(x) = \{y \in X : d(y, x) > r\}$ is open. The proof is very similar. □

Lemma. If U is an open neighbourhood of x and $x_n \to x$, then $\exists N$ such that $x_n \in U$ for all $n > N$.

Proof. Since U is open, there exists some $\delta > 0$ such that $B_\delta(x) \subseteq U$. Since $x_n \to x$, $\exists N$ such that $d(x_n, x) < \delta$ for all $n > N$. This implies that $x_n \in B_\delta(x)$ for all $n > N$. So $x_n \in U$ for all $n > N$. $\square$

Proposition. $C \subseteq X$ is a closed subset if and only if every limit point of C is an element of C.

Proof. ($\Rightarrow$) Suppose C is closed and $x_n \to x$, $x_n \in C$. We have to show that $x \in C$.

Since C is closed, $A = X \setminus C \subseteq X$ is open. Suppose the contrary that $x \notin C$. Then $x \in A$. Hence A is an open neighbourhood of x. Then by our previous lemma, we know that there is some N such that $x_n \in A$ for all $n \geq N$. So $x_N \in A$. But we know that $x_N \in C$ by assumption. This is a contradiction. So we must have $x \in C$.

($\Leftarrow$) Suppose that C is not closed. We have to find a limit point not in C.

Since C is not closed, A is not open. So $\exists x \in A$ such that $B_\delta(x) \not\subseteq A$ for all $\delta > 0$. This means that $B_\delta(x) \cap C \neq \emptyset$ for all $\delta > 0$.

So pick $x_n \in B_{\frac{1}{n}}(x) \cap C$ for each $n > 0$. Then $x_n \in C$, $d(x_n, x) = \frac{1}{n} \to 0$. So $x_n \to x$. So x is a limit point of C which is not in C. $\square$

Proposition (Characterization of continuity)**.** Let (X, d_x) and (Y, d_y) be metric spaces, and $f : X \to Y$. The following conditions are equivalent:

(i) f is continuous

(ii) If $x_n \to x$, then $f(x_n) \to f(x)$ (which is the definition of continuity)

(iii) For any closed subset $C \subseteq Y$, $f^{-1}(C)$ is closed in X.

(iv) For any open subset $U \subseteq Y$, $f^{-1}(U)$ is open in X.

(v) For any $x \in X$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Alternatively, $d_x(x, z) < \delta \Rightarrow d_y(f(x), f(z)) < \varepsilon$.

Proof.

- $1 \Leftrightarrow 2$: by definition
- $2 \Rightarrow 3$: Suppose $C \subseteq Y$ is closed. We want to show that $f^{-1}(C)$ is closed. So let $x_n \to x$, where $x_n \in f^{-1}(C)$.

  We know that $f(x_n) \to f(x)$ by (2) and $f(x_n) \in C$. So $f(x)$ is a limit point of C. Since C is closed, $f(x) \in C$. So $x \in f^{-1}(C)$. So every limit point of $f^{-1}(C)$ is in $f^{-1}(C)$. So $f^{-1}(C)$ is closed.
- $3 \Rightarrow 4$: If $U \subseteq Y$ is open, then $Y \setminus U$ is closed in Y. So $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed in X. So $f^{-1}(U) \subseteq X$ is open.
- $4 \Rightarrow 5$: Given $x \in X, \varepsilon > 0$, $B_\varepsilon(f(x))$ is open in Y. By (4), we know $f^{-1}(B_\varepsilon(f(x))) = A$ is open in X. Since $x \in A$, $\exists \delta > 0$ with $B_\delta(x) \subseteq A$. So
$$f(B_\delta(x)) \subseteq f(A) = f(f^{-1}(B_\varepsilon(f(x)))) = B_\varepsilon(f(x))$$

- $5 \Rightarrow 2$: Suppose $x_n \to x$. Given $\varepsilon > 0$, $\exists \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Since $x_n \to x$, $\exists N$ such that $x_n \in B_\delta(x)$ for all $n > N$. Then $f(x_n) \in f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ for all $n > N$. So $f(x_n) \to f(x)$. □

Lemma.

(i) $\emptyset$ and X are open subsets of X.

(ii) Suppose $V_\alpha \subseteq X$ is open for all $\alpha \in A$. Then $U = \displaystyle\bigcup_{\alpha \in A} V_\alpha$ is open in X.

(iii) If $V_1, \cdots, V_n \subseteq X$ are open, then so is $V = \displaystyle\bigcap_{i=1}^{n} V_i$.

Proof.

(i) $\emptyset$ satisfies the definition of an open subset vacuously. X is open since for any x, $B_1(x) \subseteq X$.

(ii) If $x \in U$, then $x \in V_\alpha$ for some α. Since V_α is open, there exists $\delta > 0$ such that $B_\delta(x) \subseteq V_\alpha$. So $B_\delta(x) \subseteq \displaystyle\bigcup_{\alpha \in A} V_\alpha = U$. So U is open.

(iii) If $x \in V$, then $x \in V_i$ for all $i = 1, \cdots, n$. So $\exists \delta_i > 0$ with $B_{\delta_i}(x) \subseteq V_i$. Take $\delta = \min\{\delta_1, \cdots, \delta_n\}$. So $B_\delta(x) \subseteq V_i$ for all i. So $B_\delta(x) \subseteq V$. So V is open. □

2 Topological spaces

2.1 Definitions

Lemma. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.

Proof. If $U \subseteq Z$ is open, g is continuous, then $g^{-1}(U)$ is open in Y. Since f is also continuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in X. □

Lemma. Homeomorphism is an equivalence relation.

Proof.

(i) The identity map $I_X : X \to X$ is always a homeomorphism. So $X \simeq X$.

(ii) If $f : X \to Y$ is a homeomorphism, then so is $f^{-1} : Y \to X$. So $X \simeq Y \Rightarrow Y \simeq X$.

(iii) If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then $g \circ f : X \to Z$ is a homeomorphism. So $X \simeq Y$ and $Y \simeq Z$ implies $X \simeq Z$. □

2.2 Sequences

Lemma. If X is Hausdorff, x_n is a sequence in X with $x_n \to x$ and $x_n \to x'$, then $x = x'$, i.e. limits are unique.

Proof. Suppose the contrary that $x \neq x'$. Then by definition of Hausdorff, there exist open neighbourhoods U, U' of x, x' respectively with $U \cap U' = \emptyset$.

Since $x_n \to x$ and U is a neighbourhood of x, by definition, there is some N such that whenever $n > N$, we have $x_n \in U$. Similarly, since $x_n \to x'$, there is some N' such that whenever $n > N'$, we have $x_n \in U'$.

This means that whenever $n > \max(N, N')$, we have $x_n \in U$ and $x_n \in U'$. So $x_n \in U \cap U'$. This contradicts the fact that $U \cap U' = \emptyset$.

Hence we must have $x = x'$. □

2.3 Closed sets

Lemma.

(i) If C_α is a closed subset of X for all $\alpha \in A$, then $\bigcap_{\alpha \in A} C_\alpha$ is closed in X.

(ii) If $C_1, \cdots, C_n$ are closed in X, then so is $\bigcup_{i=1}^n C_i$.

Proof.

(i) Since C_α is closed in X, $X \setminus C_\alpha$ is open in X. So $\bigcup_{\alpha \in A}(X \setminus C_\alpha) = X \setminus \bigcap_{\alpha \in A} C_\alpha$ is open. So $\bigcap_{\alpha \in A} C_\alpha$ is closed.

(ii) If C_i is closed in X, then $X \setminus C_i$ is open. So $\bigcap_{i=1}^n (X \setminus C_i) = X \setminus \bigcup_{i=1}^n C_i$ is open. So $\bigcup_{i=1}^n C_i$ is closed. □

Corollary. If X is Hausdorff and $x \in X$, then $\{x\}$ is closed in X.

Proof. For all $y \in X$, there exist open subsets U_y, V_y with $y \in U_y, x \in V_y$, $U_y \cap V_y = \emptyset$.

Let $C_y = X \setminus U_y$. Then C_y is closed, $y \notin C_y$, $x \in C_y$. So $\{x\} = \bigcap_{y \neq x} C_y$ is closed since it is an intersection of closed subsets. □

2.4 Closure and interior

2.4.1 Closure

Proposition. $\bar{A}$ is the smallest closed subset of X which contains A.

Proof. Let $K \subseteq X$ be a closed set containing A. Then $K \in \mathcal{C}_A$. So $\bar{A} = \bigcap_{C \in \mathcal{C}_A} C \subseteq K$. So $\bar{A} \subseteq K$. □

Lemma. If $C \subseteq X$ is closed, then $L(C) = C$.

Proof. Exactly the same as that for metric spaces. We will also prove a more general result very soon that implies this. □

Proposition. $L(A) \subseteq \bar{A}$.

Proof. If $A \subseteq C$, then $L(A) \subseteq L(C)$. If C is closed, then $L(C) = C$. So $C \in \mathcal{C}_A \Rightarrow L(A) \subseteq C$. So $L(A) \subseteq \bigcap_{C \in \mathcal{C}_A} C = \bar{A}$. □

Corollary. Given a subset $A \subseteq X$, if we can find some closed C such that $A \subseteq C \subseteq L(A)$, then we in fact have $C = \bar{A}$.

Proof. $C \subseteq L(A) \subseteq \bar{A} \subseteq C$, where the last step is since $\bar{A}$ is the smallest closed set containing A. So $C = L(A) = \bar{A}$. □

2.4.2 Interior

Proposition. $\text{Int}(A)$ is the largest open subset of X contained in A.

Proposition. $X \setminus \text{Int}(A) = \overline{X \setminus A}$

Proof. $U \subseteq A \Leftrightarrow (X \setminus U) \supseteq (X \setminus A)$. Also, U open in $X \Leftrightarrow X \setminus U$ is closed in X.

So the complement of the largest open subset of X contained in A will be the smallest closed subset containing $X \setminus A$. □

2.5 New topologies from old

2.5.1 Subspace topology

Proposition. The subspace topology is a topology.

Proof.

(i) Since $\emptyset$ is open in X, $\emptyset = Y \cap \emptyset$ is open in Y.

Since X is open in X, $Y = Y \cap X$ is open in Y.

(ii) If V_α is open in Y, then $V_\alpha = Y \cap U_\alpha$ for some U_α open in X. Then

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A} (Y \cap U_\alpha) = Y \cap \left(\bigcup_{\alpha \in U} U_\alpha\right).$$

Since $\bigcup U_\alpha$ is open in X, so $\bigcup V_\alpha$ is open in Y.

(iii) If V_i is open in Y, then $V_i = Y \cap U_i$ for some open $U_i \subseteq X$. Then

$$\bigcap_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (Y \cap U_i) = Y \cap \left(\bigcap_{i=1}^{n} U_i\right).$$

Since $\bigcap U_i$ is open, $\bigcap V_i$ is open. □

Proposition. If Y has the subspace topology, $f : Z \to Y$ is continuous iff $\iota \circ f : Z \to X$ is continuous.

Proof. ($\Rightarrow$) If $U \subseteq X$ is open, then $\iota^{-1}(U) = Y \cap U$ is open in Y. So ι is continuous. So if f is continuous, so is $\iota \circ f$.

($\Leftarrow$) Suppose we know that $\iota \circ f$ is continuous. Given $V \subseteq Y$ is open, we know that $V = Y \cap U = \iota^{-1}(U)$. So $f^{-1}(V) = f^{-1}(\iota^{-1}(U)) = (\iota \circ f)^{-1}(U)$ is open since $\iota \circ f$ is continuous. So f is continuous. □

2.5.2 Product topology

2.5.3 Quotient topology

3 Connectivity

3.1 Connectivity

Proposition. X is disconnected iff there exists a continuous surjective $f : X \to \{0, 1\}$ with the discrete topology.

Alternatively, X is connected iff any continuous map $f : X \to \{0, 1\}$ is constant.

Proof. ($\Rightarrow$) If A and B disconnect X, define

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

Then $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{0, 1\}) = X$, $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$ are all open. So f is continuous. Also, since A, B are non-empty, f is surjective.

($\Leftarrow$) Given $f : X \mapsto \{0, 1\}$ surjective and continuous, define $A = f^{-1}(\{0\})$, $B = f^{-1}(\{1\})$. Then A and B disconnect X. $\square$

Theorem. $[0, 1]$ is connected.

Proof. Suppose A and B disconnect $[0, 1]$. wlog, assume $1 \in B$. Since A is non-empty, $\alpha = \sup A$ exists. Then either

- $\alpha \in A$. Then $\alpha < 1$, since $1 \in B$. Since A is open, $\exists \varepsilon > 0$ with $B_\varepsilon(\alpha) \subseteq A$. So $\alpha + \frac{\varepsilon}{2} \in A$, contradicting supremality of α; or
- $\alpha \not\in A$. Then $\alpha \in B$. Since B is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(\alpha) \subseteq B$. Then $a \leq \alpha - \varepsilon$ for all $a \in A$. This contradicts α being the *least* upper bound of A.

Either option gives a contradiction. So A and B cannot exist and $[0, 1]$ is connected. $\square$

Proposition. If $f : X \to Y$ is continuous and X is connected, then $\operatorname{im} f$ is also connected.

Proof. Suppose A and B disconnect $\operatorname{im} f$. We will show that $f^{-1}(A)$ and $f^{-1}(B)$ disconnect X.

Since $A, B \subseteq \operatorname{im} f$ are open, we know that $A = \operatorname{im} f \cap A'$ and $B = \operatorname{im} f \cap B'$ for some A', B' open in Y. Then $f^{-1}(A) = f^{-1}(A')$ and $f^{-1}(B) = f^{-1}(B')$ are open in X.

Since A, B are non-empty, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty. Also, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$. Finally, $A \cup B = \operatorname{im} f$. So $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = X$.

So $f^{-1}(A)$ and $f^{-1}(B)$ disconnect X, contradicting our hypothesis. So $\operatorname{im} f$ is connected. $\square$

Theorem (Intermediate value theorem)**.** Suppose $f : X \to \mathbb{R}$ is continuous and X is connected. If $\exists x_0, x_1$ such that $f(x_0) < 0 < f(x_1)$, then $\exists x \in X$ with $f(x) = 0$.

Proof. Suppose no such x exists. Then $0 \not\in \operatorname{im} f$ while $0 > f(x_0) \in \operatorname{im} f$, $0 < f(x_1) \in \operatorname{im} f$. Then $\operatorname{im} f$ is disconnected (from our previous example), contradicting X being connected. $\square$

Corollary. If $f : [0,1] \to \mathbb{R}$ is continuous with $f(0) < 0 < f(1)$, then $\exists x \in [0,1]$ with $f(x) = 0$.

3.2 Path connectivity

Proposition. If X is path connected, then X is connected.

Proof. Let X be path connected, and let $f : X \to \{0,1\}$ be a continuous function. We want to show that f is constant.

Let $x, y \in X$. By path connectedness, there is a map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Composing with f gives a map $f \circ \gamma : [0,1] \to \{0,1\}$. Since $[0,1]$ is connected, this must be constant. In particular, $f(\gamma(0)) = f(\gamma(1))$, i.e. $f(x) = f(y)$. Since x, y were arbitrary, we know f is constant. □

Lemma. Suppose $f : X \to Y$ is a homeomorphism and $A \subseteq X$, then $f|_A : A \to f(A)$ is a homeomorphism.

Proof. Since f is a bijection, $f|_A$ is a bijection. If $U \subseteq f(A)$ is open, then $U = f(A) \cap U'$ for some U' open in Y. So $f|_A^{-1}(U) = f^{-1}(U') \cap A$ is open in A. So $f|_A$ is continuous. Similarly, we can show that $(f|_A)^{-1}$ is continuous. □

3.2.1 Higher connectivity*

3.3 Components

3.3.1 Path components

Lemma. Define $x \sim y$ if there is a path from x to y in X. Then $\sim$ is an equivalence relation.

Proof.

(i) For any $x \in X$, let $\gamma_x : [0,1] \to X$ be $\gamma(t) = x$, the constant path. Then this is a path from x to x. So $x \sim x$.

(ii) If $\gamma : [0,1] \to X$ is a path from x to y, then $\bar{\gamma} : [0,1] \to X$ by $t \mapsto \gamma(1-t)$ is a path from y to x. So $x \sim y \Rightarrow y \sim x$.

(iii) If γ_1 is a path from x to y and γ_2 is a path from y to z, then $\gamma_2 * \gamma_1$ defined by
$$t \mapsto \begin{cases} \gamma_1(2t) & t \in [0, 1/2] \\ \gamma_2(2t-1) & t \in [1/2, 1] \end{cases}$$
is a path from x to z. So $x \sim y, y \sim z \Rightarrow x \sim z$. □

3.3.2 Connected components

Proposition. Suppose $Y_\alpha \subseteq X$ is connected for all $\alpha \in T$ and that $\bigcap_{\alpha \in T} Y_\alpha \neq \emptyset$. Then $Y = \bigcup_{\alpha \in T} Y_\alpha$ is connected.

Proof. Suppose the contrary that A and B disconnect Y. Then A and B are open in Y. So $A = Y \cap A'$ and $B = Y \cap B'$, where A', B' are open in X. For any fixed α, let
$$A_\alpha = Y_\alpha \cap A = Y_\alpha \cap A', \quad B_\alpha = Y_\alpha \cap B = Y_\alpha \cap B'.$$

Then they are open in Y_α. Since $Y = A \cup B$, we have

$$Y_\alpha = Y \cap Y_\alpha = (A \cup B) \cap Y_\alpha = A_\alpha \cup B_\alpha.$$

Since $A \cap B = \emptyset$, we have

$$A_\alpha \cap B_\alpha = Y_\alpha \cap (A \cap B) = \emptyset.$$

So A_α, B_α are disjoint. So Y_α is connected but is the disjoint union of open subsets A_α, B_α.

By definition of connectivity, this can only happen if $A_\alpha = \emptyset$ or $B_\alpha = \emptyset$.

However, by assumption, $\bigcap_{\alpha \in T} Y_\alpha \neq \emptyset$. So pick $y \in \bigcap_{\alpha \in T} Y_\alpha$. Since $y \in Y$, either $y \in A$ or $y \in B$. wlog, assume $y \in A$. Then $y \in Y_\alpha$ for all α implies that $y \in A_\alpha$ for all α. So A_α is non-empty for all α. So B_α is empty for all α. So $B = \emptyset$.

So A and B did not disconnect Y after all. Contradiction. □

Lemma. If $y \in C(x)$, then $C(y) = C(x)$.

Proof. Since $y \in C(x)$ and $C(x)$ is connected, $C(x) \subseteq C(y)$. So $x \in C(y)$. Then $C(y) \subseteq C(x)$. So $C(x) = C(y)$. □

Proposition. If $U \subseteq \mathbb{R}^n$ is open and connected, then it is path-connected.

Proof. Let A be a path component of U. We first show that A is open.

Let $a \in A$. Since U is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(a) \subseteq U$. We know that $B_\varepsilon(a) \simeq \mathrm{Int}(D^n)$ is path-connected (e.g. use line segments connecting the points). Since A is a path component and $a \in A$, we must have $B_\varepsilon(a) \subseteq A$. So A is an open subset of U.

Now suppose $b \in U \setminus A$. Then since U is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(b) \subseteq U$. Since $B_\varepsilon(b)$ is path-connected, so if $B_\varepsilon(b) \cap A \neq \emptyset$, then $B_\varepsilon(b) \subseteq A$. But this implies $b \in A$, which is a contradiction. So $B_\varepsilon(b) \cap A = \emptyset$. So $B_\varepsilon(b) \subseteq U \setminus A$. Then $U \setminus A$ is open.

So $A, U \setminus A$ are disjoint open subsets of U. Since U is connected, we must have $U \setminus A$ empty (since A is not). So $U = A$ is path-connected. □

4 Compactness

4.1 Compactness

Theorem. $[0,1]$ is compact.

Proof. Suppose $\mathcal{V}$ is an open cover of $[0,1]$. Let

$$A = \{a \in [0,1] : [0,a] \text{ has a finite subcover of } \mathcal{V}\}.$$

First show that A is non-empty. Since $\mathcal{V}$ covers $[0,1]$, in particular, there is some V_0 that contains 0. So $\{0\}$ has a finite subcover V_0. So $0 \in A$.

Next we note that by definition, if $0 \leq b \leq a$ and $a \in A$, then $b \in A$.

Now let $\alpha = \sup A$. Suppose $\alpha < 1$. Then $\alpha \in [0,1]$.

Since $\mathcal{V}$ covers X, let $\alpha \in V_\alpha$. Since V_α is open, there is some ε such that $B_\varepsilon(\alpha) \subseteq V_\alpha$. By definition of α, we must have $\alpha - \varepsilon/2 \in A$. So $[0, \alpha - \varepsilon/2]$ has a finite subcover. Add V_α to that subcover to get a finite subcover of $[0, \alpha + \varepsilon/2]$. Contradiction (technically, it will be a finite subcover of $[0, \eta]$ for $\eta = \min(\alpha + \varepsilon/2, 1)$, in case $\alpha + \varepsilon/2$ gets too large).

So we must have $\alpha = \sup A = 1$.

Now we argue as before: $\exists V_1 \in \mathcal{V}$ such that $1 \in V_1$ and $\exists \varepsilon > 0$ with $(1-\varepsilon, 1] \subseteq V_1$. Since $1 - \varepsilon \in A$, there exists a finite $\mathcal{V}' \subseteq \mathcal{V}$ which covers $[0, 1-\varepsilon/2]$. Then $\mathcal{W} = \mathcal{V}' \cup \{V_1\}$ is a finite subcover of $\mathcal{V}$. □

Proposition. If X is compact and C is a closed subset of X, then C is also compact.

Proof. To prove this, given an open cover of C, we need to find a finite subcover. To do so, we need to first convert it into an open cover of X. We can do so by adding $X \setminus C$, which is open since C is closed. Then since X is compact, we can find a finite subcover of this, which we can convert back to a finite subcover of C.

Formally, suppose $\mathcal{V}$ is an open cover of C. Say $\mathcal{V} = \{V_\alpha : \alpha \in T\}$. For each α, since V_α is open in C, $V_\alpha = C \cap V'_\alpha$ for some V'_α open in X. Also, since $\bigcup_{\alpha \in T} V_a = C$, we have $\bigcup_{\alpha \in T} V'_\alpha \supseteq C$.

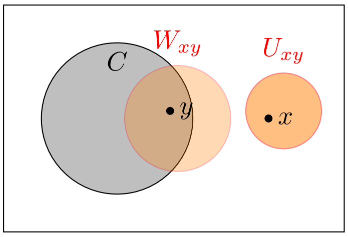
Since C is closed, $U = X \setminus C$ is open in X. So $\mathcal{W} = \{V'_\alpha : \alpha \in T\} \cup \{U\}$ is an open cover of X. Since X is compact, $\mathcal{W}$ has a finite subcover $\mathcal{W}' = \{V'_{\alpha_1}, \cdots, V'_{\alpha_n}, U\}$ (U may or may not be in there, but it doesn't matter). Now $U \cap C = \emptyset$. So $\{V_{\alpha_1}, \cdots, V_{\alpha_n}\}$ is a finite subcover of C. □

Proposition. Let X be a Hausdorff space. If $C \subseteq X$ is compact, then C is closed in X.

Proof. Let $U = X \setminus C$. We will show that U is open.

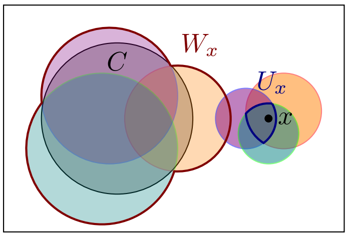
For any x, we will find a U_x such that $U_x \subseteq U$ and $x \in U_x$. Then $U = \bigcup_{x \in U} U_x$ will be open since it is a union of open sets.

To construct U_x, fix $x \in U$. Since X is Hausdorff, for each $y \in C$, $\exists U_{xy}, W_{xy}$ open neighbourhoods of x and y respectively with $U_{xy} \cap W_{xy} = \emptyset$.



Then $\mathcal{W} = \{W_{xy} \cap C : y \in C\}$ is an open cover of C. Since C is compact, there exists a finite subcover $\mathcal{W}' = \{W_{xy_1} \cap C, \cdots, W_{xy_n} \cap C\}$.

Let $U_x = \bigcap_{i=1}^n U_{xy_i}$. Then U_x is open since it is a finite intersection of open sets. To show $U_x \subseteq U$, note that $W_x = \bigcup_{i=1}^n W_{xy_i} \supseteq C$ since $\{W_{xy_i} \cap C\}$ is an open cover. We also have $W_x \cap U_x = \emptyset$. So $U_x \subseteq U$. So done.



□

Proposition. A compact metric space (X, d) is bounded.

Proof. Pick $x \in X$. Then $V = \{B_r(x) : r \in \mathbb{R}^+\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{B_{r_1}(x), \cdots, B_{r_n}(x)\}$.

Let $R = \max\{r_1, \cdots, r_n\}$. Then $d(x, y) < R$ for all $y \in X$. So for all $y, z \in X$,

$$d(y, z) \leq d(y, x) + d(x, z) < 2R$$

So X is bounded. □

Theorem (Heine-Borel)**.** $C \subseteq \mathbb{R}$ is compact iff C is closed and bounded.

Proof. Since $\mathbb{R}$ is a metric space (hence Hausdorff), C is also a metric space.

So if C is compact, C is closed in $\mathbb{R}$, and C is bounded, by our previous two propositions.

Conversely, if C is closed and bounded, then $C \subseteq [-N, N]$ for some $N \in \mathbb{R}$. Since $[-N, N] \simeq [0, 1]$ is compact, and $C = C \cap [-N, N]$ is closed in $[-N, N]$, C is compact. □

Corollary. If $A \subseteq \mathbb{R}$ is compact, $\exists \alpha \in A$ such that $\alpha \geq a$ for all $a \in A$.

Proof. Since A is compact, it is bounded. Let $\alpha = \sup A$. Then by definition, $\alpha \geq a$ for all $a \in A$. So it is enough to show that $\alpha \in A$.

Suppose $\alpha \notin A$. Then $\alpha \in \mathbb{R} \setminus A$. Since A is compact, it is closed in $\mathbb{R}$. So $\mathbb{R} \setminus A$ is open. So $\exists \varepsilon > 0$ such that $B_\varepsilon(\alpha) \subseteq \mathbb{R} \setminus A$, which implies that $a \leq \alpha - \varepsilon$ for all $a \in A$. This contradicts the assumption that $\alpha = \sup A$. So we can conclude $\alpha \in A$. □

Proposition. If $f : X \to Y$ is continuous and X is compact, then $\operatorname{im} f \subseteq Y$ is also compact.

Proof. Suppose $\mathcal{V} = \{V_\alpha : \alpha \in T\}$ is an open cover of $\operatorname{im} f$. Since V_α is open in $\operatorname{im} f$, we have $V_\alpha = \operatorname{im} f \cap V'_\alpha$, where V'_α is open in Y. Then

$$W_\alpha = f^{-1}(V_\alpha) = f^{-1}(V'_\alpha)$$

is open in X. If $x \in X$ then $f(x)$ is in V_α for some α, so $x \in W_\alpha$. Thus $\mathcal{W} = \{W_\alpha : \alpha \in T\}$ is an open cover of X.

Since X is compact, there's a finite subcover $\{W_{\alpha_1}, \cdots, W_{\alpha_n}\}$ of $\mathcal{W}$.

Since $V_\alpha \subseteq \operatorname{im} f$, $f(W_\alpha) = f(f^{-1}(V_\alpha)) = V_\alpha$. So

$$\{V_{\alpha_1}, \cdots, V_{\alpha_n}\}$$

is a finite subcover of $\mathcal{V}$. □

Theorem (Maximum value theorem)**.** If $f : X \to \mathbb{R}$ is continuous and X is compact, then $\exists x \in X$ such that $f(x) \geq f(y)$ for all $y \in X$.

Proof. Since X is compact, $\operatorname{im} f$ is compact. Let $\alpha = \max\{\operatorname{im} f\}$. Then $\alpha \in \operatorname{im} f$. So $\exists x \in X$ with $f(x) = \alpha$. Then by definition $f(x) \geq f(y)$ for all $y \in X$. □

Corollary. If $f : [0, 1] \to \mathbb{R}$ is continuous, then $\exists x \in [0, 1]$ such that $f(x) \geq f(y)$ for all $y \in [0, 1]$

Proof. $[0, 1]$ is compact. □

4.2 Products and quotients

4.2.1 Products

Theorem. If X and Y are compact, then so is $X \times Y$.

Proof. First consider the special type of open cover $\mathcal{V}$ of $X \times Y$ such that every $U \in \mathcal{V}$ has the form $U = V \times W$, where $V \subseteq X$ and $W \subseteq Y$ are open.

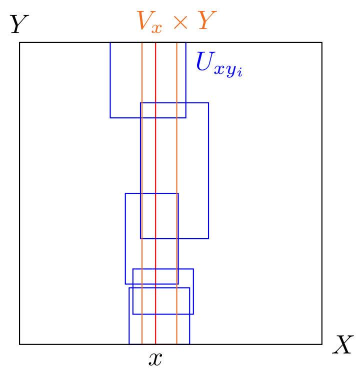
For every $(x, y) \in X \times Y$, there is $U_{xy} \in \mathcal{V}$ with $(x, y) \in U_{xy}$. Write

$$U_{xy} = V_{xy} \times W_{xy},$$

where $V_{xy} \subseteq X$, $W_{xy} \subseteq Y$ are open, $x \in V_{xy}, y \in W_{xy}$.

Fix $x \in X$. Then $\mathcal{W}_x = \{W_{xy} : y \in Y\}$ is an open cover of Y. Since Y is compact, there is a finite subcover $\{W_{xy_1}, \cdots, W_{xy_n}\}$.

Then $V_x = \bigcap_{i=1}^n V_{xy_i}$ is a finite intersection of open sets. So V_x is open in X. Moreover, $\mathcal{V}_x = \{U_{xy_1}, \cdots, U_{xy_n}\}$ covers $V_x \times Y$.



Now $\mathcal{O} = \{V_x : x \in X\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{V_{x_1}, \cdots, V_{x_m}\}$. Then $\mathcal{V}' = \bigcup_{i=1}^m \mathcal{V}_{x_i}$ is a finite subset of $\mathcal{V}$, which covers all of $X \times Y$.

In the general, case, suppose $\mathcal{V}$ is an open cover of $X \times Y$. For each $(x, y) \in X \times Y$, $\exists U_{xy} \in \mathcal{V}$ with $(x, y) \in U_{xy}$. Since U_{xy} is open, $\exists V_{xy} \subseteq X, W_{xy} \subseteq Y$ open with $V_{xy} \times W_{xy} \subseteq U_{xy}$ and $x \in V_{xy}$, $y \in W_{xy}$.

Then $\mathcal{Q} = \{V_{xy} \times W_{xy} : (x, y) \in (X, Y)\}$ is an open cover of $X \times Y$ of the type we already considered above. So it has a finite subcover $\{V_{x_1y_1} \times W_{x_1y_1}, \cdots, V_{x_ny_n} \times W_{x_ny_n}\}$. Now $V_{x_iy_i} \times W_{x_iy_i} \subseteq U_{x_iy_i}$. So $\{U_{x_1y_1}, \cdots, U_{x_ny_n}\}$ is a finite subcover of $X \times Y$. □

Corollary (Heine-Borel in $\mathbb{R}^n$)**.** $C \subseteq \mathbb{R}^n$ is compact iff C is closed and bounded.

Proof. If C is bounded, $C \subseteq [-N, N]^n$ for some $N \in \mathbb{R}$, which is compact. The rest of the proof is exactly the same as for $n = 1$. □

4.2.2 Quotients

Proposition. Suppose $f : X \to Y$ is a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. We show that f^{-1} is continuous. To do this, it suffices to show $(f^{-1})^{-1}(C)$ is closed in Y whenever C is closed in X. By hypothesis, f is a bijection . So $(f^{-1})^{-1}(C) = f(C)$.

Supposed C is closed in X. Since X is compact, C is compact. Since f is continuous, $f(C) = (\operatorname{im} f|_C)$ is compact. Since Y is Hausdorff and $f(C) \subseteq Y$ is compact, $f(C)$ is closed. □

Corollary. Suppose $f : X/\sim \to Y$ is a bijection, X is compact, Y is Hausdorff, and $f \circ \pi$ is continuous, then f is a homeomorphism.

Proof. Since X is compact and $\pi : X \mapsto X/\sim$ is continuous, $\operatorname{im} \pi \subseteq X/\sim$ is compact. Since $f \circ \pi$ is continuous, f is continuous. So we can apply the proposition. □

4.3 Sequential compactness

Lemma. Let (x_n) be a sequence in a metric space (X, d) and $x \in X$. Then (x_n) has a subsequence converging to x iff for every $\varepsilon > 0$, $x_n \in B_\varepsilon(x)$ for infinitely many n $(*)$.

Proof. If $(x_{n_i}) \to x$, then for every ε, we can find I such that $i > I$ implies $x_{n_i} \in B_\varepsilon(x)$ by definition of convergence. So $(*)$ holds.

Now suppose $(*)$ holds. We will construct a sequence $x_{n_i} \to x$ inductively. Take $n_0 = 0$. Suppose we have defined $x_{n_0}, \cdots, x_{n_{i-1}}$.

By hypothesis, $x_n \in B_{1/i}(x)$ for infinitely many n. Take n_i to be smallest such n with $n_i > n_{i-1}$.

Then $d(x_{n_i}, x) < \frac{1}{i}$ implies that $x_{n_i} \to x$. □

Theorem. If (X, d) is a compact *metric space*, then X is sequentially compact.

Proof. Suppose x_n is a sequence in X with no convergent subsequence. Then for any $y \in X$, there is no subsequence converging to y. By lemma, there exists $\varepsilon > 0$ such that $x_n \in B_\varepsilon(y)$ for only finitely many n.

Let $U_y = B_\varepsilon(y)$. Now $\mathcal{V} = \{U_y : y \in X\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{U_{y_1}, \cdots, U_{y_m}\}$. Then $x_n \in \bigcup_{i=1}^m U_{y_i} = X$ for only finitely many n. This is nonsense, since $x_n \in X$ for all n!

So x_n must have a convergent subsequence. $\square$

4.4 Completeness

Proposition. If X is a compact metric space, then X is complete.

Proof. Let x_n be a Cauchy sequence in X. Since X is sequentially compact, there is a convergent subsequence $x_{n_i} \to x$. We will show that $x_n \to x$.

Given $\varepsilon > 0$, pick N such that $d(x_n, x_m) < \varepsilon/2$ for $n, m \geq N$. Pick I such that $n_I \geq N$ and $d(x_{n_i}, x) < \varepsilon/2$ for all $i > I$. Then for $n \geq n_I$, $d(x_n, x) \leq d(x_n, x_{n_I}) + d(x_{n_I}, x) < \varepsilon$. So $x_n \to x$. $\square$

Corollary. $\mathbb{R}^n$ is complete.

Proof. If $(x_n) \subseteq \mathbb{R}^n$ is Cauchy, then $(x_n) \subseteq \bar{B}_R(0)$ for some R, and $\bar{B}_R(0)$ is compact. So it converges. $\square$