Part IB — Metric and Topological Spaces
Theorems

Based on lectures by J. Rasmussen
Notes taken by Dexter Chua

Easter 2015

Metrics
Definition and examples. Limits and continuity. Open sets and neighbourhoods. Characterizing limits and continuity using neighbourhoods and open sets. [3]

Topology
Definition of a topology. Metric topologies. Further examples. Neighbourhoods, closed sets, convergence and continuity. Hausdorff spaces. Homeomorphisms. Topological and non-topological properties. Completeness. Subspace, quotient and product topologies. [3]

Connectedness
Definition using open sets and integer-valued functions. Examples, including intervals. Components. The continuous image of a connected space is connected. Path-connectedness. Path-connected spaces are connected but not conversely. Connected open sets in Euclidean space are path-connected. [3]

Compactness
Definition using open covers. Examples: finite sets and [0, 1]. Closed subsets of compact spaces are compact. Compact subsets of a Hausdorff space must be closed. The compact subsets of the real line. Continuous images of compact sets are compact. Quotient spaces. Continuous real-valued functions on a compact space are bounded and attain their bounds. The product of two compact spaces is compact. The compact subsets of Euclidean space. Sequential compactness. [3]

Contents

0 Introduction

1 Metric spaces

1.1 Definitions

Proposition. If (X, d) is a metric space, (x_n) is a sequence in X such that $x_n \to x$, $x_n \to x'$, then $x = x'$.

1.2 Examples of metric spaces

1.3 Norms

Lemma. If $\|\cdot\|$ is a norm on V, then

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$

defines a metric on V.

Lemma. Let $f \in C[0, 1]$ satisfy $f(x) \geq 0$ for all $x \in [0, 1]$. If $f(x)$ is not constantly 0, then $\int_0^1 f(x)\, \mathrm{d}x > 0$.

Theorem (Cauchy-Schwarz inequality)**.** If $\langle \cdot\,, \cdot \rangle$ is an inner product, then

$$\langle \mathbf{v}, \mathbf{w}\rangle^2 \leq \langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{w}, \mathbf{w}\rangle.$$

Lemma. If $\langle \cdot\,, \cdot \rangle$ is an inner product on V, then

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$

is a norm.

1.4 Open and closed subsets

Lemma. The open ball $B_r(x) \subseteq X$ is an open subset, and the closed ball $\bar{B}_r(x) \subseteq X$ is a closed subset.

Lemma. If U is an open neighbourhood of x and $x_n \to x$, then $\exists N$ such that $x_n \in U$ for all $n > N$.

Proposition. $C \subseteq X$ is a closed subset if and only if every limit point of C is an element of C.

Proposition (Characterization of continuity)**.** Let (X, d_x) and (Y, d_y) be metric spaces, and $f : X \to Y$. The following conditions are equivalent:

(i) f is continuous

(ii) If $x_n \to x$, then $f(x_n) \to f(x)$ (which is the definition of continuity)

(iii) For any closed subset $C \subseteq Y$, $f^{-1}(C)$ is closed in X.

(iv) For any open subset $U \subseteq Y$, $f^{-1}(U)$ is open in X.

(v) For any $x \in X$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Alternatively, $d_x(x, z) < \delta \Rightarrow d_y(f(x), f(z)) < \varepsilon$.

Lemma.

(i) $\emptyset$ and X are open subsets of X.

(ii) Suppose $V_\alpha \subseteq X$ is open for all $\alpha \in A$. Then $U = \bigcup_{\alpha \in A} V_\alpha$ is open in X.

(iii) If $V_1, \cdots, V_n \subseteq X$ are open, then so is $V = \bigcap_{i=1}^{n} V_i$.

2 Topological spaces

2.1 Definitions

Lemma. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.

Lemma. Homeomorphism is an equivalence relation.

2.2 Sequences

Lemma. If X is Hausdorff, x_n is a sequence in X with $x_n \to x$ and $x_n \to x'$, then $x = x'$, i.e. limits are unique.

2.3 Closed sets

Lemma.

(i) If C_α is a closed subset of X for all $\alpha \in A$, then $\bigcap_{\alpha \in A} C_\alpha$ is closed in X.

(ii) If $C_1, \cdots, C_n$ are closed in X, then so is $\bigcup_{i=1}^n C_i$.

Corollary. If X is Hausdorff and $x \in X$, then $\{x\}$ is closed in X.

2.4 Closure and interior

2.4.1 Closure

Proposition. $\bar{A}$ is the smallest closed subset of X which contains A.

Lemma. If $C \subseteq X$ is closed, then $L(C) = C$.

Proposition. $L(A) \subseteq \bar{A}$.

Corollary. Given a subset $A \subseteq X$, if we can find some closed C such that $A \subseteq C \subseteq L(A)$, then we in fact have $C = \bar{A}$.

2.4.2 Interior

Proposition. $\mathrm{Int}(A)$ is the largest open subset of X contained in A.

Proposition. $X \setminus \mathrm{Int}(A) = \overline{X \setminus A}$

2.5 New topologies from old

2.5.1 Subspace topology

Proposition. The subspace topology is a topology.

Proposition. If Y has the subspace topology, $f : Z \to Y$ is continuous iff $\iota \circ f : Z \to X$ is continuous.

2.5.2 Product topology

2.5.3 Quotient topology

3 Connectivity

3.1 Connectivity

Proposition. X is disconnected iff there exists a continuous surjective $f : X \to \{0, 1\}$ with the discrete topology.

Alternatively, X is connected iff any continuous map $f : X \to \{0, 1\}$ is constant.

Theorem. $[0, 1]$ is connected.

Proposition. If $f : X \to Y$ is continuous and X is connected, then $\operatorname{im} f$ is also connected.

Theorem (Intermediate value theorem)**.** Suppose $f : X \to \mathbb{R}$ is continuous and X is connected. If $\exists x_0, x_1$ such that $f(x_0) < 0 < f(x_1)$, then $\exists x \in X$ with $f(x) = 0$.

Corollary. If $f : [0, 1] \to \mathbb{R}$ is continuous with $f(0) < 0 < f(1)$, then $\exists x \in [0, 1]$ with $f(x) = 0$.

3.2 Path connectivity

Proposition. If X is path connected, then X is connected.

Lemma. Suppose $f : X \to Y$ is a homeomorphism and $A \subseteq X$, then $f|_A : A \to f(A)$ is a homeomorphism.

3.2.1 Higher connectivity*

3.3 Components

3.3.1 Path components

Lemma. Define $x \sim y$ if there is a path from x to y in X. Then $\sim$ is an equivalence relation.

3.3.2 Connected components

Proposition. Suppose $Y_\alpha \subseteq X$ is connected for all $\alpha \in T$ and that $\bigcap_{\alpha \in T} Y_\alpha \neq \emptyset$. Then $Y = \bigcup_{\alpha \in T} Y_\alpha$ is connected.

Lemma. If $y \in C(x)$, then $C(y) = C(x)$.

Proposition. If $U \subseteq \mathbb{R}^n$ is open and connected, then it is path-connected.

4 Compactness

4.1 Compactness

Theorem. $[0, 1]$ is compact.

Proposition. If X is compact and C is a closed subset of X, then C is also compact.

Proposition. Let X be a Hausdorff space. If $C \subseteq X$ is compact, then C is closed in X.

Proposition. A compact metric space (X, d) is bounded.

Theorem (Heine-Borel)**.** $C \subseteq \mathbb{R}$ is compact iff C is closed and bounded.

Corollary. If $A \subseteq \mathbb{R}$ is compact, $\exists \alpha \in A$ such that $\alpha \geq a$ for all $a \in A$.

Proposition. If $f : X \to Y$ is continuous and X is compact, then $\operatorname{im} f \subseteq Y$ is also compact.

Theorem (Maximum value theorem)**.** If $f : X \to \mathbb{R}$ is continuous and X is compact, then $\exists x \in X$ such that $f(x) \geq f(y)$ for all $y \in X$.

Corollary. If $f : [0, 1] \to \mathbb{R}$ is continuous, then $\exists x \in [0, 1]$ such that $f(x) \geq f(y)$ for all $y \in [0, 1]$

4.2 Products and quotients

4.2.1 Products

Theorem. If X and Y are compact, then so is $X \times Y$.

Corollary (Heine-Borel in $\mathbb{R}^n$)**.** $C \subseteq \mathbb{R}^n$ is compact iff C is closed and bounded.

4.2.2 Quotients

Proposition. Suppose $f : X \to Y$ is a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Corollary. Suppose $f : X/{\sim} \to Y$ is a bijection, X is compact, Y is Hausdorff, and $f \circ \pi$ is continuous, then f is a homeomorphism.

4.3 Sequential compactness

Lemma. Let (x_n) be a sequence in a metric space (X, d) and $x \in X$. Then (x_n) has a subsequence converging to x iff for every $\varepsilon > 0$, $x_n \in B_\varepsilon(x)$ for infinitely many n $(*)$.

Theorem. If (X, d) is a compact *metric space*, then X is sequentially compact.

4.4 Completeness

Proposition. If X is a compact metric space, then X is complete.

Corollary. $\mathbb{R}^n$ is complete.