Metric and Topological Spaces **Easter 2015**

Example Sheet 1

1. Show that the sequence $2015, 20015, 200015, 2000015\ldots$ converges in the 2-adic metric on $\mathbb{Z}$.

2. Determine whether the following subsets $A \subset \mathbb{R}^2$ are open, closed, or neither:

   (a) $A = \{(x,y) \mid x < 0\} \cup \{(x,y) \mid x > 0, y > 1/x\}$

   (b) $A = \{(x, \sin(1/x) \mid x > 0\} \cup \{(0,y) \mid y \in [-1,1]\}$

   (c) $A = \{(x,y) \mid x \in \mathbb{Q}, x = y^n \text{ for some positive integer } n\}$.

3. Show that the maps $f, g : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x + y$ and $f(x,y) = xy$ are continuous with respect to the usual topology on $\mathbb{R}$. Let X be $\mathbb{R}$ equipped with the topology whose open sets are intervals of the form (a, ∞). Are the maps $f, g : X \times X \to X$ continuous?

4. Let $\mathbf{C}^1[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is differentiable and } f' \text{ is continuous}\}$. For $f \in \mathbf{C}^1[0,1]$, define

$$\|f\|_{1,1} = \int_0^1 (|f(x)| + |f'(x)|)\, dx.$$

   Show that $\|\cdot\|_{1,1}$ defines a norm on $\mathbf{C}^1[0,1]$. If a sequence (f_n) converges with respect to this norm, show that it also converges with respect to the uniform norm. Give an example to show that the converse statement does not hold.

5. Let $d : X \times X \to \mathbb{R}$ be a function which satisfies all the axioms for a metric space except for the requirement that $d(x,y) = 0 \Leftrightarrow x = y$. For $x, y \in X$, define $x \sim y$ if $d(x,y) = 0$. Show that $\sim$ is an equivalence relation on X, and that d induces a metric on the quotient $X/\sim$.

6. Find a closed $A_1 \subset \mathbb{R}$ (with the usual topology) so that $\overline{\text{Int}(A_1)} \neq A_1$ and an open $A_2 \subset \mathbb{R}$ so that $\text{Int}(\overline{A_2}) \neq A_2$.

7. Let $f : X \to Y$ be a map of topological spaces. Show that f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$. Deduce that if f is surjective and continuous, the image of a dense set in X is dense in Y.

8. Suppose X is a topological space and $Z \subset Y \subset X$. If Y is dense in X and Z is dense in Y (with the subspace topology), must Z be dense in X?

9. Define a topology on $\mathbb{R}$ by declaring the closed subsets to be those which are i) closed in the usual topology and ii) either bounded or all of $\mathbb{R}$. Show that this is a topology, that all points of $\mathbb{R}$ are closed with respect to it, but that the topology is not Hausdorff.

10. The *diagonal* in $X \times X$ is the set $\Delta_X = \{(x, x) \mid x \in X\}$. If X is a Hausdorff topological space, show that Δ_X is a closed subset of $X \times X$.

11. Exhibit a countable basis for the usual topology on $\mathbb{R}$.

12. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus. Let $L \subset \mathbb{R}^2$ be a line of the form $y = \alpha x$, where α is irrational, and let $\pi(L)$ be its image in T^2. What are the closure and interior of $\pi(L)$?

13. Let $A = \{(0, 0, 1), (0, 0, -1)\} \subset S^2$. Let $B \subset T^2$ be the image of $\mathbb{R} \times 0 \subset \mathbb{R}^2$, where we view $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Show that the quotient spaces S^2/A and T^2/B are homeomorphic.

14. Let $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be a function which satisfies all the axioms for a norm except possibly the triangle inequality. Let $B = \{\mathbf{v} \in \mathbb{R}^2 \mid \|\mathbf{v}\| \leq 1\}$. Show that $\|\cdot\|$ is a norm if and only if B is a convex subset of $\mathbb{R}^2$. (That is, if $\mathbf{v}_1, \mathbf{v}_2 \in B$, then $t\mathbf{v}_1 + (1-t)\mathbf{v}_2 \in B$ for $t \in [0, 1]$.) For $r \in (0, \infty)$, let $\|\mathbf{v}\|_r = (|v_1|^r + |v_2|^r)^{1/r}$. Use calculus to sketch B for different values of r. Deduce that $\|\cdot\|_r$ is a norm for $1 \leq r < \infty$, but not for $0 < r < 1$.

15. Let D^2 be the closed unit disk in $\mathbb{R}^2$, and let X be the complement of two disjoint open disks in D^2. Let Y be the complement of a small open disk in T^2 (viewed as $\mathbb{R}^2/\mathbb{Z}^2$). Is X homeomorphic to Y? Is $X \times [0, 1]$ homeomorphic to $Y \times [0, 1]$? (No formal proof is required, but try to give some geometric justification.)

16. Show that the set of piecewise linear functions is dense in $\mathbf{C}[0, 1]$ with the uniform metric. By considering piecewise linear functions where each linear piece is given by an expression with rational coefficients, deduce that $\mathbf{C}[0, 1]$ has a countable dense subset.

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Example Sheet 2

1. Which of the following subsets of $\mathbb{R}^2$ are a) connected b) path connected?
   (a) $B_1((1,0)) \cup B_1((-1,0))$
   (b) $\overline{B}_1((1,0)) \cup B_1((-1,0))$
   (c) $\{(x,y) \,|\, y = 0 \text{ or } x/y \in \mathbb{Q}\}$
   (d) $\{(x,y) \,|\, y = 0 \text{ or } x/y \in \mathbb{Q}\} - \{(0,0)\}$

2. Suppose that X is connected, and that $f : X \to Y$ is a locally constant map; *i.e.* for every $x \in X$, there is an open neighborhood U of x such that $f(y) = f(x)$ for all $y \in U$. Show that f is constant.

3. Show that the product of two connected spaces is connected.

4. Show there is no continuous injective map $f : \mathbb{R}^2 \to \mathbb{R}$.

5. Show that $\mathbb{R}^2$ with the topology induced by the British rail metric is not homeomorphic to $\mathbb{R}^2$ with the topology induced by the Euclidean metric.

6. Let X be a topological space. If A is a connected subspace of X, show that $\overline{A}$ is also connected. Deduce that any component of X is a closed subset of X.

7. (a) If $f : [0,1] \to [0,1]$ is continuous, show there is some $x \in [0,1]$ with $f(x) = x$.
   (b) Suppose $f : [0,1] \to \mathbb{R}$ is continuous and has $f(0) = f(1)$. For each integer $n > 1$, show that there is some $x \in [0,1]$ with $f(x) = f(x + \frac{1}{n})$.

8. A standard chair (four legs, feet are the vertices of a square) is placed on an uneven floor (modeled by the graph of a continuous function $z = g(x,y)$.) By rotating the chair about its center, show that it is always possible to find a position where all four feet are on the floor.

9. Is there an infinite compact subset of $\mathbb{Q}$?

10. If $A \subset \mathbb{R}^n$ is not compact, show there is a continuous function $f : A \to \mathbb{R}$ which is not bounded.

11. If X is a topological space, its *one point compactification* X^+ is defined as follows. As a set, X^+ is the union of X with an additional point ∞. A subset $U \subset X^+$ is open if either

(a) $\infty \notin U$ and U is an open subset of X

(b) $\infty \in U$ and $X^+ - U$ is a compact, closed subset of X.

Show that X^+ is a compact topological space. If $X = \mathbb{R}^n$, show that $X^+ \simeq S^n$.

12. Suppose that X is a compact Hausdorff space, and that C_1 and C_2 are disjoint closed subsets of X. Show that there exist open subsets $U_1, U_2 \subset X$ such that $C_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$.

13. Let (X, d) be a metric space. A complete metric space (X', d') is said to be a *completion* of (X, d) if a) $X \subset X'$ and $d'|_{X \times X} = d$ and b) X is dense in X'.

    (a) Suppose that (Y, d_Y) is a complete metric space and that $f : X \to Y$ is an *isometric embedding*, *i.e.* $d_Y(f(x_1), f(x_2)) = d(x_1, x_2)$. Show that f extends to an isometric embedding $f' : X' \to Y$.

    (b) Deduce that any two completions of X are *isometric*, *i.e.* related by an bijective isometric embedding.

14. If p is a prime number, let $\mathbb{Z}_p$ be the space of sequences $(x_n)_{n \geq 0}$ in $\mathbb{Z}/p\mathbb{Z}$, equipped with the metric $d((x_n), (y_n)) = p^{-k}$, where k is the smallest value of n such that $x_n \neq y_n$.

    (a) Find an isometric embedding of $f : (\mathbb{Z}, d_p) \to \mathbb{Z}_p$, where d_p is the p-adic metric. Show that $\mathbb{Z}_p$ is a completion of the image of f. The set $\mathbb{Z}_p$ is called the p-adic numbers.

    (b) Show that $\mathbb{Z}_p$ is compact and totally disconnected.

    (c) Show that the maps $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f(x, y) = x + y$, $g(x, y) = xy$ extend to continuous maps $f', g' : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$.

    (d) Let a be an integer which is relatively prime to p and assume $p > 2$. Show that the equation $x^2 = a$ has a solution in $\mathbb{Z}_p$ if and only if it has a solution in $\mathbb{Z}/p\mathbb{Z}$.

15. Show that $C[0, 1]$ equipped with the uniform metric is complete.

16. Define a norm $\|\cdot\|_{\infty,\infty}$ on $C^1[0, 1]$ by $\|f\|_{\infty,\infty} = \max\{\|f\|_\infty, \|f'\|_\infty\}$. Let $B = \overline{B}_1(0)$ be the closed unit ball in this norm. Show that any sequence (f_n) in B has a subsequence which converges with respect to the uniform norm. (Hint: first find a subsequence (f_{n_i}) such that $f_{n_i}(x)$ converges for all $x \in \mathbb{Q} \cap [0, 1]$.) Deduce that the closure of B in $(C[0, 1], d_\infty)$ is compact.

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