

Part IB — Metric and Topological Spaces

Definitions

Based on lectures by J. Rasmussen

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Easter 2015

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Metrics

Definition and examples. Limits and continuity. Open sets and neighbourhoods. Characterizing limits and continuity using neighbourhoods and open sets. [3]

Topology

Definition of a topology. Metric topologies. Further examples. Neighbourhoods, closed sets, convergence and continuity. Hausdorff spaces. Homeomorphisms. Topological and non-topological properties. Completeness. Subspace, quotient and product topologies. [3]

Connectedness

Definition using open sets and integer-valued functions. Examples, including intervals. Components. The continuous image of a connected space is connected. Path-connectedness. Path-connected spaces are connected but not conversely. Connected open sets in Euclidean space are path-connected. [3]

Compactness

Definition using open covers. Examples: finite sets and $[0, 1]$. Closed subsets of compact spaces are compact. Compact subsets of a Hausdorff space must be closed. The compact subsets of the real line. Continuous images of compact sets are compact. Quotient spaces. Continuous real-valued functions on a compact space are bounded and attain their bounds. The product of two compact spaces is compact. The compact subsets of Euclidean space. Sequential compactness. [3]

Contents

0	Introduction	3
1	Metric spaces	4
1.1	Definitions	4
1.2	Examples of metric spaces	4
1.3	Norms	4
1.4	Open and closed subsets	5
2	Topological spaces	6
2.1	Definitions	6
2.2	Sequences	6
2.3	Closed sets	6
2.4	Closure and interior	7
2.4.1	Closure	7
2.4.2	Interior	7
2.5	New topologies from old	7
2.5.1	Subspace topology	7
2.5.2	Product topology	7
2.5.3	Quotient topology	7
3	Connectivity	8
3.1	Connectivity	8
3.2	Path connectivity	8
3.2.1	Higher connectivity*	8
3.3	Components	8
3.3.1	Path components	8
3.3.2	Connected components	8
4	Compactness	9
4.1	Compactness	9
4.2	Products and quotients	9
4.2.1	Products	9
4.2.2	Quotients	9
4.3	Sequential compactness	9
4.4	Completeness	9

0 Introduction

1 Metric spaces

1.1 Definitions

Definition (Metric space). A *metric space* is a pair (X, d_X) where X is a set (the *space*) and d_X is a function $d_X : X \times X \rightarrow \mathbb{R}$ (the *metric*) such that for all x, y, z ,

$$- d_X(x, y) \geq 0 \quad (\text{non-negativity})$$

$$- d_X(x, y) = 0 \text{ iff } x = y \quad (\text{identity of indiscernibles})$$

$$- d_X(x, y) = d_X(y, x) \quad (\text{symmetry})$$

$$- d_X(x, z) \leq d_X(x, y) + d_X(y, z) \quad (\text{triangle inequality})$$

Definition (Metric subspace). Let (X, d_X) be a metric space, and $Y \subseteq X$. Then (Y, d_Y) is a metric space, where $d_Y(a, b) = d_X(a, b)$, and said to be a *subspace* of X .

Definition (Convergent sequences). Let (x_n) be a sequence in a metric space (X, d_X) . We say (x_n) *converges* to $x \in X$, written $x_n \rightarrow x$, if $d(x_n, x) \rightarrow 0$ (as a real sequence). Equivalently,

$$(\forall \varepsilon > 0)(\exists N)(\forall n > N) d(x_n, x) < \varepsilon.$$

Definition (Continuous function). Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \rightarrow Y$. We say f is *continuous* if $f(x_n) \rightarrow f(x)$ (in Y) whenever $x_n \rightarrow x$ (in X).

1.2 Examples of metric spaces

1.3 Norms

Definition (Norm). Let V be a real vector space. A *norm* on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}$ such that

$$- \|\mathbf{v}\| \geq 0 \text{ for all } \mathbf{v} \in V$$

$$- \|\mathbf{v}\| = 0 \text{ if and only if } \mathbf{v} = \mathbf{0}.$$

$$- \|\lambda \mathbf{v}\| = |\lambda| \|\mathbf{v}\|$$

$$- \|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

Definition (Inner product). Let V be a real vector space. An *inner product* on V is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ such that

$$(i) \langle \mathbf{v}, \mathbf{v} \rangle \geq 0 \text{ for all } \mathbf{v} \in V$$

$$(ii) \langle \mathbf{v}, \mathbf{v} \rangle = 0 \text{ if and only if } \mathbf{v} = \mathbf{0}.$$

$$(iii) \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle.$$

$$(iv) \langle \mathbf{v}_1 + \lambda \mathbf{v}_2, \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{w} \rangle + \lambda \langle \mathbf{v}_2, \mathbf{w} \rangle.$$

1.4 Open and closed subsets

Definition (Open and closed balls). Let (X, d) be a metric space. For any $x \in X$, $r \in \mathbb{R}$,

$$B_r(x) = \{y \in X : d(y, x) < r\}$$

is the *open ball* centered at x .

$$\bar{B}_r(x) = \{y \in X : d(y, x) \leq r\}$$

is the *closed ball* centered at x .

Definition (Open subset). $U \subseteq X$ is an *open subset* if for every $x \in U$, $\exists \delta > 0$ such that $B_\delta(x) \subseteq U$.

$C \subseteq X$ is a *closed subset* if $X \setminus C \subseteq X$ is open.

Definition (Open neighborhood). If $x \in X$, an *open neighborhood* of x is an open $U \subseteq X$ with $x \in U$.

Definition (Limit point). Let $A \subseteq X$. Then $x \in X$ is a *limit point* of A if there is a sequence $x_n \rightarrow x$ such that $x_n \in A$ for all n .

2 Topological spaces

2.1 Definitions

Definition (Topological space). A *topological space* is a set X (the space) together with a set $\mathcal{U} \subseteq \mathbb{P}(X)$ (the topology) such that:

- (i) $\emptyset, X \in \mathcal{U}$
- (ii) If $V_\alpha \in \mathcal{U}$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} V_\alpha \in \mathcal{U}$.
- (iii) If $V_1, \dots, V_n \in \mathcal{U}$, then $\bigcap_{i=1}^n V_i \in \mathcal{U}$.

The elements of X are the *points*, and the elements of \mathcal{U} are the open subsets of X .

Definition (Induced topology). Let (X, d) be a metric space. Then the topology *induced by d* is the set of all open sets of X under d .

Definition (Continuous function). Let $f : X \rightarrow Y$ be a map of topological spaces. Then f is *continuous* if $f^{-1}(U)$ is open in X whenever U is open in Y .

Definition (Homeomorphism). $f : X \rightarrow Y$ is a *homeomorphism* if

- (i) f is a bijection
- (ii) Both f and f^{-1} are continuous

Equivalently, f is a bijection and $U \subseteq X$ is open iff $f(U) \subseteq Y$ is open.

Two spaces are *homeomorphic* if there exists a homeomorphism between them, and we write $X \simeq Y$.

2.2 Sequences

Definition (Open neighbourhood). An *open neighbourhood* of $x \in X$ is an open set $U \subseteq X$ with $x \in U$.

Definition (Convergent sequence). A sequence $x_n \rightarrow x$ if for every open neighbourhood U of x , $\exists N$ such that $x_n \in U$ for all $n > N$.

Definition (Hausdorff space). A topological space X is *Hausdorff* if for all $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exist open neighbourhoods U_1 of x_1 , U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$.

2.3 Closed sets

Definition (Closed sets). $C \subseteq X$ is *closed* if $X \setminus C$ is an open subset of X .

2.4 Closure and interior

2.4.1 Closure

Definition. Let X be a topological space and $A \subseteq X$. Define

$$\mathcal{C}_A = \{C \subseteq X : A \subseteq C \text{ and } C \text{ is closed in } X\}$$

Then the *closure* of A in X is

$$\bar{A} = \bigcap_{C \in \mathcal{C}_A} C.$$

Definition (Limit point). A *limit point* of A is an $x \in X$ such that there is a sequence $x_n \rightarrow x$ with $x_n \in A$ for all n .

Definition (Dense subset). $A \subseteq X$ is *dense* in X if $\bar{A} = X$.

2.4.2 Interior

Definition (Interior). Let $A \subseteq X$, and let

$$\mathcal{O}_A = \{U \subseteq X : U \subseteq A, U \text{ is open in } X\}.$$

The *interior* of A is

$$\text{Int}(A) = \bigcup_{U \in \mathcal{O}_A} U.$$

2.5 New topologies from old

2.5.1 Subspace topology

Definition (Subspace topology). Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is given by: V is an open subset of Y if there is some U open in X such that $V = Y \cap U$.

2.5.2 Product topology

Definition (Product topology). Let X and Y be topological spaces. The *product topology* on $X \times Y$ is given by:

$U \subseteq X \times Y$ is open if: for every $(x, y) \in U$, there exist $V_x \subseteq X, W_y \subseteq Y$ open neighbourhoods of x and y such that $V_x \times W_y \subseteq U$.

Definition (Basis). Let \mathcal{U} be a topology on X . A subset $\mathcal{B} \subseteq \mathcal{U}$ is a *basis* if “ $U \in \mathcal{U}$ iff U is a union of sets in \mathcal{B} ”.

2.5.3 Quotient topology

Definition (Quotient topology). If X is a topological space, the *quotient topology* on X/\sim is given by: U is open in X/\sim if $\pi^{-1}(U)$ is open in X .

3 Connectivity

3.1 Connectivity

Definition (Connected space). A topological space X is *disconnected* if X can be written as $A \cup B$, where A and B are disjoint, non-empty open subsets of X . We say A and B *disconnect* X .

A space is *connected* if it is not disconnected.

3.2 Path connectivity

Definition (Path). Let X be a topological space, and $x_0, x_1 \in X$. Then a *path* from x_0 to x_1 is a continuous function $\gamma : [0, 1] \mapsto X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

Definition (Path connectivity). A topological space X is *path connected* if for all points $x_0, x_1 \in X$, there is a path from x_0 to x_1 .

3.2.1 Higher connectivity*

Definition (n -connectedness). X is *n -connected* if for any $k \leq n$, any continuous $f : S^k \rightarrow X$ extends to a continuous $F : D^{k+1} \rightarrow X$ such that $F|_{S^k} = f$.

3.3 Components

3.3.1 Path components

Definition (Path components). Equivalence classes of the relation “ $x \sim y$ if there is a path from x to y ” are *path components* of X .

3.3.2 Connected components

Definition (Connected component). If $x \in X$, define

$$\mathcal{C}(x) = \{A \subseteq X : x \in A \text{ and } A \text{ is connected}\}.$$

Then $C(x) = \bigcup_{A \in \mathcal{C}(x)} A$ is the *connected component* of x .

4 Compactness

4.1 Compactness

Definition (Open cover). Let $\mathcal{U} \subseteq \mathbb{P}(X)$ be a topology on X . An *open cover* of X is a subset $\mathcal{V} \subseteq \mathcal{U}$ such that

$$\bigcup_{V \in \mathcal{V}} V = X.$$

We say \mathcal{V} *covers* X .

If $\mathcal{V}' \subseteq \mathcal{V}$, and \mathcal{V}' covers X , then we say \mathcal{V}' is a *subcover* of \mathcal{V} .

Definition (Compact space). A topological space X is *compact* if every open cover \mathcal{V} of X has a finite subcover $\mathcal{V}' = \{V_1, \dots, V_n\} \subseteq \mathcal{V}$.

Definition (Bounded metric space). A metric space (X, d) is *bounded* if there exists $M \in \mathbb{R}$ such that $d(x, y) \leq M$ for all $x, y \in X$.

4.2 Products and quotients

4.2.1 Products

4.2.2 Quotients

4.3 Sequential compactness

Definition (Sequential compactness). A topological space X is *sequentially compact* if every sequence (x_n) in X has a convergent subsequence (that converges to a point in X !).

4.4 Completeness

Definition (Cauchy sequence). Let (X, d) be a metric space. A sequence (x_n) in X is *Cauchy* if for every $\varepsilon > 0$, $\exists N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

Definition (Complete space). A metric space (X, d) is *complete* if every Cauchy sequence in X converges to a limit in X .