Part IA — Vectors and Matrices
Theorems

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Complex numbers
Review of complex numbers, including complex conjugate, inverse, modulus, argument and Argand diagram. Informal treatment of complex logarithm, n-th roots and complex powers. de Moivre's theorem. [2]

Vectors
Review of elementary algebra of vectors in $\mathbb{R}^3$, including scalar product. Brief discussion of vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$; scalar product and the Cauchy-Schwarz inequality. Concepts of linear span, linear independence, subspaces, basis and dimension.

Suffix notation: including summation convention, δ_{ij} and ε_{ijk}. Vector product and triple product: definition and geometrical interpretation. Solution of linear vector equations. Applications of vectors to geometry, including equations of lines, planes and spheres. [5]

Matrices
Elementary algebra of 3×3 matrices, including determinants. Extension to $n \times n$ complex matrices. Trace, determinant, non-singular matrices and inverses. Matrices as linear transformations; examples of geometrical actions including rotations, reflections, dilations, shears; kernel and image. [4]

Simultaneous linear equations: matrix formulation; existence and uniqueness of solutions, geometric interpretation; Gaussian elimination. [3]

Symmetric, anti-symmetric, orthogonal, hermitian and unitary matrices. Decomposition of a general matrix into isotropic, symmetric trace-free and antisymmetric parts. [1]

Eigenvalues and Eigenvectors
Eigenvalues and eigenvectors; geometric significance. [2]

Proof that eigenvalues of hermitian matrix are real, and that distinct eigenvalues give an orthogonal basis of eigenvectors. The effect of a general change of basis (similarity transformations). Diagonalization of general matrices: sufficient conditions; examples of matrices that cannot be diagonalized. Canonical forms for 2×2 matrices. [5]

Discussion of quadratic forms, including change of basis. Classification of conics, cartesian and polar forms. [1]

Rotation matrices and Lorentz transformations as transformation groups. [1]

Contents

0 Introduction

1 Complex numbers

1.1 Basic properties

Proposition. $z\bar{z} = a^2 + b^2 = |z|^2$.

Proposition. $z^{-1} = \bar{z}/|z|^2$.

Theorem (Triangle inequality)**.** For all $z_1, z_2 \in \mathbb{C}$, we have

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Alternatively, we have $|z_1 - z_2| \geq ||z_1| - |z_2||$.

1.2 Complex exponential function

Lemma.

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} = \sum_{r=0}^{\infty} \sum_{m=0}^{r} a_{r-m,m}$$

Theorem. $\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$

Theorem. $e^{iz} = \cos z + i \sin z$.

1.3 Roots of unity

Proposition. If $\omega = \exp\left(\frac{2\pi i}{n}\right)$, then $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$

1.4 Complex logarithm and power

1.5 De Moivre's theorem

Theorem (De Moivre's theorem)**.**

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n.$$

1.6 Lines and circles in $\mathbb{C}$

Theorem (Equation of straight line)**.** The equation of a straight line through z_0 and parallel to w is given by

$$z\bar{w} - \bar{z}w = z_0\bar{w} - \bar{z}_0 w.$$

Theorem. The general equation of a circle with center $c \in \mathbb{C}$ and radius $\rho \in \mathbb{R}^+$ can be given by

$$z\bar{z} - \bar{c}z - c\bar{z} = \rho^2 - c\bar{c}.$$

2 Vectors

2.1 Definition and basic properties

2.2 Scalar product

2.2.1 Geometric picture ($\mathbb{R}^2$ and $\mathbb{R}^3$ only)

2.2.2 General algebraic definition

2.3 Cauchy-Schwarz inequality

Theorem (Cauchy-Schwarz inequality)**.** For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|.$$

Corollary (Triangle inequality)**.**

$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|.$$

2.4 Vector product

Proposition.

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) \times (b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}) \\
&= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} + \cdots \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned}$$

2.5 Scalar triple product

Proposition. If a parallelepiped has sides represented by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ that form a right-handed system, then the volume of the parallelepiped is given by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

Theorem. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

2.6 Spanning sets and bases

2.6.1 2D space

Theorem. The coefficients λ, μ are unique.

2.6.2 3D space

Theorem. If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are non-coplanar, i.e. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \neq 0$, then they form a basis of $\mathbb{R}^3$.

2.6.3 $\mathbb{R}^n$ space

2.6.4 $\mathbb{C}^n$ space

2.7 Vector subspaces

2.8 Suffix notation

Proposition. $(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$

Theorem. $\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$

Proposition.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$

Theorem (Vector triple product)**.**

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Proposition. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})$.

2.9 Geometry

2.9.1 Lines

Theorem. The equation of a straight line through $\mathbf{a}$ and parallel to $\mathbf{t}$ is

$$(\mathbf{x} - \mathbf{a}) \times \mathbf{t} = \mathbf{0} \text{ or } \mathbf{x} \times \mathbf{t} = \mathbf{a} \times \mathbf{t}.$$

2.9.2 Plane

Theorem. The equation of a plane through $\mathbf{b}$ with normal $\mathbf{n}$ is given by

$$\mathbf{x} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n}.$$

2.10 Vector equations

3 Linear maps

3.1 Examples

3.1.1 Rotation in $\mathbb{R}^3$

3.1.2 Reflection in $\mathbb{R}^3$

3.2 Linear Maps

Theorem. Consider a linear map $f : U \to V$, where U, V are vector spaces. Then $\mathrm{im}(f)$ is a subspace of V, and $\ker(f)$ is a subspace of U.

3.3 Rank and nullity

Theorem (Rank-nullity theorem)**.** For a linear map $f : U \to V$,

$$r(f) + n(f) = \dim(U).$$

3.4 Matrices

3.4.1 Examples

3.4.2 Matrix Algebra

Proposition.

(i) $(A^T)^T = A$.

(ii) If $\mathbf{x}$ is a column vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\mathbf{x}^T$ is a row vector $(x_1\ x_2 \cdots x_n)$.

(iii) $(AB)^T = B^T A^T$ since $(AB)^T_{ij} = (AB)_{ji} = A_{jk}B_{ki} = B_{ki}A_{jk}$
$= (B^T)_{ik}(A^T)_{kj} = (B^T A^T)_{ij}$.

Proposition. $\mathrm{tr}(BC) = \mathrm{tr}(CB)$

3.4.3 Decomposition of an $n \times n$ matrix

3.4.4 Matrix inverse

Proposition. $(AB)^{-1} = B^{-1}A^{-1}$

3.5 Determinants

3.5.1 Permutations

Proposition. Any q-cycle can be written as a product of 2-cycles.

Proposition.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3.5.2 Properties of determinants

Proposition. $\det(A) = \det(A^T)$.

Proposition. If matrix B is formed by multiplying every element in a single row of A by a scalar λ, then $\det(B) = \lambda \det(A)$. Consequently, $\det(\lambda A) = \lambda^n \det(A)$.

Proposition. If 2 rows (or 2 columns) of A are identical, the determinant is 0.

Proposition. If 2 rows or 2 columns of a matrix are linearly dependent, then the determinant is zero.

Proposition. Given a matrix A, if B is a matrix obtained by adding a multiple of a column (or row) of A to another column (or row) of A, then $\det A = \det B$.

Corollary. Swapping two rows or columns of a matrix negates the determinant.

Proposition. $\det(AB) = \det(A)\det(B)$.

Corollary. If A is orthogonal, $\det A = \pm 1$.

Corollary. If U is unitary, $|\det U| = 1$.

Proposition. In $\mathbb{R}^3$, orthogonal matrices represent either a rotation ($\det = 1$) or a reflection ($\det = -1$).

3.5.3 Minors and Cofactors

Theorem (Laplace expansion formula)**.** For any particular fixed i,

$$\det A = \sum_{j=1}^{n} A_{ji}\Delta_{ji}.$$

4 Matrices and linear equations

4.1 Simple example, 2×2

4.2 Inverse of an $n \times n$ matrix

Lemma. $\sum A_{ik}\Delta_{jk} = \delta_{ij} \det A$.

Theorem. If $\det A \neq 0$, then A^{-1} exists and is given by

$$(A^{-1})_{ij} = \frac{\Delta_{ji}}{\det A}.$$

4.3 Homogeneous and inhomogeneous equations

4.3.1 Gaussian elimination

4.4 Matrix rank

Theorem. The column rank and row rank are equal for any $m \times n$ matrix.

4.5 Homogeneous problem $A\mathbf{x} = \mathbf{0}$

4.5.1 Geometrical interpretation

4.5.2 Linear mapping view of $A\mathbf{x} = \mathbf{0}$

4.6 General solution of $A\mathbf{x} = \mathbf{d}$

5 Eigenvalues and eigenvectors

5.1 Preliminaries and definitions

Theorem (Fundamental theorem of algebra)**.** Let $p(z)$ be a polynomial of degree $m \geq 1$, i.e.

$$p(z) = \sum_{j=0}^{m} c_j z^j,$$

where $c_j \in \mathbb{C}$ and $c_m \neq 0$.

Then $p(z) = 0$ has precisely m (not necessarily distinct) roots in the complex plane, accounting for multiplicity.

Theorem. λ is an eigenvalue of A iff

$$\det(A - \lambda I) = 0.$$

5.2 Linearly independent eigenvectors

Theorem. Suppose $n \times n$ matrix A has *distinct* eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then the corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are linearly independent.

5.3 Transformation matrices

5.3.1 Transformation law for vectors

Theorem. Denote vector as $\mathbf{u}$ with respect to $\{\mathbf{e}_i\}$ and $\tilde{\mathbf{u}}$ with respect to $\{\tilde{\mathbf{e}_i}\}$. Then

$$\mathbf{u} = P\tilde{\mathbf{u}} \text{ and } \tilde{\mathbf{u}} = P^{-1}\mathbf{u}$$

5.3.2 Transformation law for matrix

Theorem.

$$\tilde{A} = P^{-1}AP.$$

5.4 Similar matrices

Proposition. Similar matrices have the following properties:

(i) Similar matrices have the same determinant.

(ii) Similar matrices have the same trace.

(iii) Similar matrices have the same characteristic polynomial.

5.5 Diagonalizable matrices

Theorem. Let $\lambda_1, \lambda_2, \cdots, \lambda_r$, with $r \leq n$ be the distinct eigenvalues of A. Let $B_1, B_2, \cdots B_r$ be the bases of the eigenspaces $E_{\lambda_1}, E_{\lambda_2}, \cdots, E_{\lambda_r}$ correspondingly. Then the set $B = \bigcup_{i=1}^{r} B_i$ is linearly independent.

Proposition. A is diagonalizable iff all its eigenvalues have zero defect.

5.6 Canonical (Jordan normal) form

Theorem. Any 2×2 complex matrix A is similar to exactly one of

$$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}, \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$$

Proposition. (Without proof) The canonical form, or Jordan normal form, exists for any $n \times n$ matrix A. Specifically, there exists a similarity transform such that A is similar to a matrix to $\tilde{A}$ that satisfies the following properties:

(i) $\tilde{A}_{\alpha\alpha} = \lambda_\alpha$, i.e. the diagonal composes of the eigenvalues.

(ii) $\tilde{A}_{\alpha,\alpha+1} = 0$ or 1.

(iii) $\tilde{A}_{ij} = 0$ otherwise.

5.7 Cayley-Hamilton Theorem

Theorem (Cayley-Hamilton theorem)**.** Every $n \times n$ complex matrix satisfies its own characteristic equation.

5.8 Eigenvalues and eigenvectors of a Hermitian matrix

5.8.1 Eigenvalues and eigenvectors

Theorem. The eigenvalues of a Hermitian matrix H are real.

Theorem. The eigenvectors of a Hermitian matrix H corresponding to distinct eigenvalues are orthogonal.

5.8.2 Gram-Schmidt orthogonalization (non-examinable)

5.8.3 Unitary transformation

5.8.4 Diagonalization of $n \times n$ Hermitian matrices

Theorem. An $n \times n$ Hermitian matrix has precisely n orthogonal eigenvectors.

5.8.5 Normal matrices

Proposition.

(i) If λ is an eigenvalue of N, then λ^* is an eigenvalue of $N^\dagger$.

(ii) The eigenvectors of distinct eigenvalues are orthogonal.

(iii) A normal matrix can always be diagonalized with an orthonormal basis of eigenvectors.

6 Quadratic forms and conics

Theorem. Hermitian forms are real.

6.1 Quadrics and conics

6.1.1 Quadrics

6.1.2 Conic sections ($n = 2$)

6.2 Focus-directrix property

7 Transformation groups

7.1 Groups of orthogonal matrices

Proposition. The set of all $n \times n$ orthogonal matrices P forms a group under matrix multiplication.

7.2 Length preserving matrices

Theorem. Let $P \in O(n)$. Then the following are equivalent:

(i) P is orthogonal

(ii) $|P\mathbf{x}| = |\mathbf{x}|$

(iii) $(P\mathbf{x})^T(P\mathbf{y}) = \mathbf{x}^T\mathbf{y}$, i.e. $(P\mathbf{x}) \cdot (P\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

(iv) If $(\mathbf{v}_1, \mathbf{v_2}, \cdots, \mathbf{v}_n)$ are orthonormal, so are $(P\mathbf{v}_1, P\mathbf{v}_2, \cdots, P\mathbf{v}_n)$

(v) The columns of P are orthonormal.

7.3 Lorentz transformations