

Part IA — Vectors and Matrices

Theorems

Based on lectures by N. Peake

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Complex numbers

Review of complex numbers, including complex conjugate, inverse, modulus, argument and Argand diagram. Informal treatment of complex logarithm, n -th roots and complex powers. de Moivre's theorem. [2]

Vectors

Review of elementary algebra of vectors in \mathbb{R}^3 , including scalar product. Brief discussion of vectors in \mathbb{R}^n and \mathbb{C}^n ; scalar product and the Cauchy-Schwarz inequality. Concepts of linear span, linear independence, subspaces, basis and dimension.

Suffix notation: including summation convention, δ_{ij} and ε_{ijk} . Vector product and triple product: definition and geometrical interpretation. Solution of linear vector equations. Applications of vectors to geometry, including equations of lines, planes and spheres. [5]

Matrices

Elementary algebra of 3×3 matrices, including determinants. Extension to $n \times n$ complex matrices. Trace, determinant, non-singular matrices and inverses. Matrices as linear transformations; examples of geometrical actions including rotations, reflections, dilations, shears; kernel and image. [4]

Simultaneous linear equations: matrix formulation; existence and uniqueness of solutions, geometric interpretation; Gaussian elimination. [3]

Symmetric, anti-symmetric, orthogonal, hermitian and unitary matrices. Decomposition of a general matrix into isotropic, symmetric trace-free and antisymmetric parts. [1]

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors; geometric significance. [2]

Proof that eigenvalues of hermitian matrix are real, and that distinct eigenvalues give an orthogonal basis of eigenvectors. The effect of a general change of basis (similarity transformations). Diagonalization of general matrices: sufficient conditions; examples of matrices that cannot be diagonalized. Canonical forms for 2×2 matrices. [5]

Discussion of quadratic forms, including change of basis. Classification of conics, cartesian and polar forms. [1]

Rotation matrices and Lorentz transformations as transformation groups. [1]

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0 Introduction

1 Complex numbers

1.1 Basic properties

Proposition. $z\bar{z} = a^2 + b^2 = |z|^2$.

Proposition. $z^{-1} = \bar{z}/|z|^2$.

Theorem (Triangle inequality). For all $z_1, z_2 \in \mathbb{C}$, we have

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Alternatively, we have $|z_1 - z_2| \geq ||z_1| - |z_2||$.

1.2 Complex exponential function

Lemma.

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} = \sum_{r=0}^{\infty} \sum_{m=0}^r a_{r-m,m}$$

Theorem. $\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$

Theorem. $e^{iz} = \cos z + i \sin z$.

1.3 Roots of unity

Proposition. If $\omega = \exp\left(\frac{2\pi i}{n}\right)$, then $1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0$

1.4 Complex logarithm and power

1.5 De Moivre's theorem

Theorem (De Moivre's theorem).

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n.$$

1.6 Lines and circles in \mathbb{C}

Theorem (Equation of straight line). The equation of a straight line through z_0 and parallel to w is given by

$$z\bar{w} - \bar{z}w = z_0\bar{w} - \bar{z}_0w.$$

Theorem. The general equation of a circle with center $c \in \mathbb{C}$ and radius $\rho \in \mathbb{R}^+$ can be given by

$$z\bar{z} - \bar{c}z - c\bar{z} = \rho^2 - c\bar{c}.$$

2 Vectors

2.1 Definition and basic properties

2.2 Scalar product

2.2.1 Geometric picture (\mathbb{R}^2 and \mathbb{R}^3 only)

2.2.2 General algebraic definition

2.3 Cauchy-Schwarz inequality

Theorem (Cauchy-Schwarz inequality). For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|.$$

Corollary (Triangle inequality).

$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|.$$

2.4 Vector product

Proposition.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) \times (b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}) \\ &= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} + \dots \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned}$$

2.5 Scalar triple product

Proposition. If a parallelepiped has sides represented by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ that form a right-handed system, then the volume of the parallelepiped is given by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

Theorem. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

2.6 Spanning sets and bases

2.6.1 2D space

Theorem. The coefficients λ, μ are unique.

2.6.2 3D space

Theorem. If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are non-coplanar, i.e. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \neq 0$, then they form a basis of \mathbb{R}^3 .

2.6.3 \mathbb{R}^n space**2.6.4** \mathbb{C}^n space**2.7** Vector subspaces**2.8** Suffix notation**Proposition.** $(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$ **Theorem.** $\varepsilon_{ijk} \varepsilon_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}$ **Proposition.**

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$

Theorem (Vector triple product).

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Proposition. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})$.**2.9** Geometry**2.9.1** Lines**Theorem.** The equation of a straight line through \mathbf{a} and parallel to \mathbf{t} is

$$(\mathbf{x} - \mathbf{a}) \times \mathbf{t} = \mathbf{0} \text{ or } \mathbf{x} \times \mathbf{t} = \mathbf{a} \times \mathbf{t}.$$

2.9.2 Plane**Theorem.** The equation of a plane through \mathbf{b} with normal \mathbf{n} is given by

$$\mathbf{x} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n}.$$

2.10 Vector equations

3 Linear maps

3.1 Examples

3.1.1 Rotation in \mathbb{R}^3

3.1.2 Reflection in \mathbb{R}^3

3.2 Linear Maps

Theorem. Consider a linear map $f : U \rightarrow V$, where U, V are vector spaces. Then $\text{im}(f)$ is a subspace of V , and $\text{ker}(f)$ is a subspace of U .

3.3 Rank and nullity

Theorem (Rank-nullity theorem). For a linear map $f : U \rightarrow V$,

$$r(f) + n(f) = \dim(U).$$

3.4 Matrices

3.4.1 Examples

3.4.2 Matrix Algebra

Proposition.

(i) $(A^T)^T = A$.

(ii) If \mathbf{x} is a column vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, \mathbf{x}^T is a row vector $(x_1 \ x_2 \ \cdots \ x_n)$.

(iii) $(AB)^T = B^T A^T$ since $(AB)^T_{ij} = (AB)_{ji} = A_{jk} B_{ki} = B_{ki} A_{jk} = (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}$.

Proposition. $\text{tr}(BC) = \text{tr}(CB)$

3.4.3 Decomposition of an $n \times n$ matrix

3.4.4 Matrix inverse

Proposition. $(AB)^{-1} = B^{-1}A^{-1}$

3.5 Determinants

3.5.1 Permutations

Proposition. Any q -cycle can be written as a product of 2-cycles.

Proposition.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3.5.2 Properties of determinants

Proposition. $\det(A) = \det(A^T)$.

Proposition. If matrix B is formed by multiplying every element in a single row of A by a scalar λ , then $\det(B) = \lambda \det(A)$. Consequently, $\det(\lambda A) = \lambda^n \det(A)$.

Proposition. If 2 rows (or 2 columns) of A are identical, the determinant is 0.

Proposition. If 2 rows or 2 columns of a matrix are linearly dependent, then the determinant is zero.

Proposition. Given a matrix A , if B is a matrix obtained by adding a multiple of a column (or row) of A to another column (or row) of A , then $\det A = \det B$.

Corollary. Swapping two rows or columns of a matrix negates the determinant.

Proposition. $\det(AB) = \det(A) \det(B)$.

Corollary. If A is orthogonal, $\det A = \pm 1$.

Corollary. If U is unitary, $|\det U| = 1$.

Proposition. In \mathbb{R}^3 , orthogonal matrices represent either a rotation ($\det = 1$) or a reflection ($\det = -1$).

3.5.3 Minors and Cofactors

Theorem (Laplace expansion formula). For any particular fixed i ,

$$\det A = \sum_{j=1}^n A_{ji} \Delta_{ji}.$$

4 Matrices and linear equations

4.1 Simple example, 2×2

4.2 Inverse of an $n \times n$ matrix

Lemma. $\sum A_{ik} \Delta_{jk} = \delta_{ij} \det A$.

Theorem. If $\det A \neq 0$, then A^{-1} exists and is given by

$$(A^{-1})_{ij} = \frac{\Delta_{ji}}{\det A}.$$

4.3 Homogeneous and inhomogeneous equations

4.3.1 Gaussian elimination

4.4 Matrix rank

Theorem. The column rank and row rank are equal for any $m \times n$ matrix.

4.5 Homogeneous problem $Ax = 0$

4.5.1 Geometrical interpretation

4.5.2 Linear mapping view of $Ax = 0$

4.6 General solution of $Ax = d$

5 Eigenvalues and eigenvectors

5.1 Preliminaries and definitions

Theorem (Fundamental theorem of algebra). Let $p(z)$ be a polynomial of degree $m \geq 1$, i.e.

$$p(z) = \sum_{j=0}^m c_j z^j,$$

where $c_j \in \mathbb{C}$ and $c_m \neq 0$.

Then $p(z) = 0$ has precisely m (not necessarily distinct) roots in the complex plane, accounting for multiplicity.

Theorem. λ is an eigenvalue of A iff

$$\det(A - \lambda I) = 0.$$

5.2 Linearly independent eigenvectors

Theorem. Suppose $n \times n$ matrix A has *distinct* eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then the corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are linearly independent.

5.3 Transformation matrices

5.3.1 Transformation law for vectors

Theorem. Denote vector as \mathbf{u} with respect to $\{\mathbf{e}_i\}$ and $\tilde{\mathbf{u}}$ with respect to $\{\tilde{\mathbf{e}}_i\}$. Then

$$\mathbf{u} = P\tilde{\mathbf{u}} \text{ and } \tilde{\mathbf{u}} = P^{-1}\mathbf{u}$$

5.3.2 Transformation law for matrix

Theorem.

$$\tilde{A} = P^{-1}AP.$$

5.4 Similar matrices

Proposition. Similar matrices have the following properties:

- (i) Similar matrices have the same determinant.
- (ii) Similar matrices have the same trace.
- (iii) Similar matrices have the same characteristic polynomial.

5.5 Diagonalizable matrices

Theorem. Let $\lambda_1, \lambda_2, \dots, \lambda_r$, with $r \leq n$ be the distinct eigenvalues of A . Let B_1, B_2, \dots, B_r be the bases of the eigenspaces $E_{\lambda_1}, E_{\lambda_2}, \dots, E_{\lambda_r}$ correspondingly.

Then the set $B = \bigcup_{i=1}^r B_i$ is linearly independent.

Proposition. A is diagonalizable iff all its eigenvalues have zero defect.

5.6 Canonical (Jordan normal) form

Theorem. Any 2×2 complex matrix A is similar to exactly one of

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Proposition. (Without proof) The canonical form, or Jordan normal form, exists for any $n \times n$ matrix A . Specifically, there exists a similarity transform such that A is similar to a matrix \tilde{A} that satisfies the following properties:

- (i) $\tilde{A}_{\alpha\alpha} = \lambda_\alpha$, i.e. the diagonal composes of the eigenvalues.
- (ii) $\tilde{A}_{\alpha,\alpha+1} = 0$ or 1 .
- (iii) $\tilde{A}_{ij} = 0$ otherwise.

5.7 Cayley-Hamilton Theorem

Theorem (Cayley-Hamilton theorem). Every $n \times n$ complex matrix satisfies its own characteristic equation.

5.8 Eigenvalues and eigenvectors of a Hermitian matrix

5.8.1 Eigenvalues and eigenvectors

Theorem. The eigenvalues of a Hermitian matrix H are real.

Theorem. The eigenvectors of a Hermitian matrix H corresponding to distinct eigenvalues are orthogonal.

5.8.2 Gram-Schmidt orthogonalization (non-examinable)

5.8.3 Unitary transformation

5.8.4 Diagonalization of $n \times n$ Hermitian matrices

Theorem. An $n \times n$ Hermitian matrix has precisely n orthogonal eigenvectors.

5.8.5 Normal matrices

Proposition.

- (i) If λ is an eigenvalue of N , then λ^* is an eigenvalue of N^\dagger .
- (ii) The eigenvectors of distinct eigenvalues are orthogonal.
- (iii) A normal matrix can always be diagonalized with an orthonormal basis of eigenvectors.

6 Quadratic forms and conics

Theorem. Hermitian forms are real.

6.1 Quadrics and conics

6.1.1 Quadrics

6.1.2 Conic sections ($n = 2$)

6.2 Focus-directrix property

7 Transformation groups

7.1 Groups of orthogonal matrices

Proposition. The set of all $n \times n$ orthogonal matrices P forms a group under matrix multiplication.

7.2 Length preserving matrices

Theorem. Let $P \in O(n)$. Then the following are equivalent:

- (i) P is orthogonal
- (ii) $|P\mathbf{x}| = |\mathbf{x}|$
- (iii) $(P\mathbf{x})^T(P\mathbf{y}) = \mathbf{x}^T\mathbf{y}$, i.e. $(P\mathbf{x}) \cdot (P\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.
- (iv) If $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ are orthonormal, so are $(P\mathbf{v}_1, P\mathbf{v}_2, \dots, P\mathbf{v}_n)$
- (v) The columns of P are orthonormal.

7.3 Lorentz transformations